Nonlinear Systems

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Public Version

April 23, 2024

Contents

Abbreviations

The following notation will be employed throughout this text:

$\mathbb{N}$: Set of all positive integers
$\mathbb{Z}$: Set of all integers
$\mathbb{Q}$: Set of all rational numbers
$\mathbb{R}$: Set of all real numbers, i.e. the real line
$\mathbb{C}$: Set of all complex numbers
$\mathbb{C}^-$: Set of all complex numbers with a (strictly) negative real part
$\overline{\mathbb{C}^-}$: Set of all complex numbers with a non-positive real part
$\mathbb{C}^0$: Set of all purely imaginary numbers
$\mathbb{C}^+$: Set of all complex numbers with a (strictly) positive real part
$\overline{\mathbb{C}^+}$: Set of all complex numbers with a non-negative real part
$\in$: Is an element of (Is contained in)
$\forall$: For each (for all)
$\exists$: There exists
$\exists!$: There exists a unique
$\exists?$: Does there exist
$\ni$: Such that
$A \Rightarrow B$: A implies B
$B \Leftarrow A$: B implies A
$A \Leftrightarrow B$: A and B are equivalent
I_n: Identity matrix of dimension $n \times n$
O_n: Zero matrix of dimension $n \times n$
$\circ$: Composition of Functions
$1_{\mathcal{X}}$: Identity map from $\mathcal{X}$ to $\mathcal{X}$
$S_1 \subset S_2$: The set S_1 is a subset of the set S_2
$\mathcal{W} \leq \mathcal{V}$: The vector space $\mathcal{W}$ is a subspace of the vector space $\mathcal{V}$
$\mathcal{W} \oplus \mathcal{V}$: The direct sum of $\mathcal{W}$ and $\mathcal{V}$.
$\mathcal{W} \overset{\perp}{\oplus} \mathcal{V}$: The orthogonal direct sum of $\mathcal{W}$ and $\mathcal{V}$.
$|\cdot|$: Norm of a vector
$\|\cdot\|$: Norm of a matrix or operator
$X(s)$: Unilateral Laplace transform of $x(t)$
(If the time-domain argument is capitalized, e.g. $X(t)$, a hat is used, e.g. $\hat{X}(s)$).
$u_{st}(t)$: Unit step function

LTI: Linear time-invariant
LTV: Linear time-variant
SISO: Single-input-single-output
MIMO: Multiple-input-multiple-output

January 22, 2019

Chapter 1

Linear Systems and Math Preliminaries

1.1 Linear Systems

We begin by presenting a brief history of control theory, as summarized in this chart:

Age	Tools	Examples
Pre-industrial Era ($\sim$ 1600)	Art vs. Science	Water clocks
Industrial Era (1600-1900)	ODEs, PDEs, Stability Theory Routh Criteria, Lyapunov Theory	Steam regulators, Windmills
Classical Control λ (1900-1950)	Root locus, Bode plot, Nyquist plot	Telephones, Amplifiers Bomb deployment
Modern Era (1950 $\sim$)	State-space methods	Navigation

The main purposes of controllers are to (1) stabilize a system, and (2) improve system performance. There are many types of controllers; we list a few below in increasing order of sophistication and performance:

1. Undergraduate—PID
2. Graduate—LQR, Extremum-stacking, state/output feedback, Loop-shaping, MPC
3. Nonlinear—Sliding-mode, Feedback linearization.

A brief review of key definitions and concepts in linear systems is given below.

Definition 1.1 (**State**). *$x(t) \in \mathbb{R}^n$ is a **state** for the system Σ if the initial state $x(t_0)$, and input sequence $u_{[t_0,t]}$ are sufficient to uniquely determine the output sequence $y_{[t_0,t]}$, for each $t \geq t_0$.*

Definition 1.2 (**Linear vs. Nonlinear systems**).

1. *A non-linear system, without loss of generality, has the form:*

$$\Sigma_{n,l} : \begin{cases} \dot{x} = f(x,u), \\ y = h(x,u) \end{cases}$$

   where f, h are not necessarily linear in the tuple (x,u).

2. *A linear system has the form:*

$$\Sigma_{n,l} : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases}$$

   where A, B, C, D are time-independent matrices of appropriate dimensions.

Definition 1.3. *An equilibrium point (x_{eq}, u_{eq}) of a system is a point satisfying:*

$$\dot{x}\Big|_{(x_{eq},u_{eq})} = F(x_{eq}, u_{eq}) = 0$$

Definition 1.4 (Linearization). *A nonlinear system $\Sigma_{nl} : \dot{x} = f(x,u)$ can be linearized about its equilibrium point (assumed to be $(x_{eq}, u_{eq}) = (0,0)$ via Taylor expansion on (x,u) about (x_{eq}, u_{eq}):*

$$\begin{aligned} \dot{x} &= F(x,u) \\ &= F(x_{eq}, u_{eq}) + \underbrace{\frac{\partial F}{\partial x}(x_{eq}, u_{eq})}_{\equiv A} \cdot (x - x_{eq}) + \underbrace{\frac{\partial F}{\partial u}(x_{eq}, u_{eq})}_{\equiv B} \cdot (u - u_{eq}) + h.o.t. \\ &\approx Ax + Bu, \end{aligned}$$

where $F(x_{eq}, u_{eq}) = 0$ by definition of the equilibrium point $(x_{eq}, u_{eq}) = (0,0)$, and h.o.t. stands for "higher-order terms."

January 24, 2019

Next, we will review controllability, stability, stabilizability, and feedback control. First, let us examine the equilibrium point of a simple linear time-invariant (LTI) system. This will motivate subsequent discussions.

Example. The set of all equilibrium points of the LTI system $\Sigma : \dot{x} = Ax$ is $N(A)$, the null space of A, i.e. x is an equilibrium point of Σ if and only if $x \in N(A)$. This is because:

$$\begin{aligned} &x \text{ is an equilibrium point} \\ \Rightarrow &\dot{x} = Ax = 0 \\ \Rightarrow &x \in N(A) \end{aligned}$$

We now turn to the concept of controllability. Consider first the following definitions and results.

Definition 1.5 (**Piecewise Continuity**). *The function* $f(x,t) : \mathbb{R}^n \times \overline{\mathbb{R}^+} \to \mathbb{R}^n$ *is said to be* ***piecewise continuous in*** t *if, in any closed and bounded interval:*

1. $f(x,\cdot) : \overline{\mathbb{R}^+} \to \mathbb{R}^n$ *is continuous except at a finite number of points.*

2. *All discontinuities of* f *in the interval are simple (jump) discontinuities, i.e. the left and right limits exist and differ by a finite amount.*

Note (**Notation**). Let $x(t,t_0,x_0,u_{[t,t_0]})$ denote the trajectory (solution) of a system Σ at time t, given $x(t_0) = x_0$, and control input $u_{[t_0,t]}$, as defined on the time interval $[t_0,t]$.

Theorem 1.6 (**Solution to LTI systems**). *The trajectory of the LTI system* $\Sigma : \dot{x} = Ax + Bu$ *is given by:*

$$x(t,t_0,x_0,u_{[t_0,t]}) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}B\,u(\tau)\,d\tau$$

Definition 1.7 (**Steering**). *Let* $(U,\Sigma,\mathcal{Y},s,r)$ *be a dynamical system representation, and let* t_0,t_1 *be given with* $t_0 < t_1$*. The input* $u_{[t_0,t_1]}(\cdot)$ ***steers*** (x_0,t_0) ***to*** (x_1,t_1) *if:*

$$x_1 = x(t_1,t_0,x_0,u_{[t_0,t_1]})$$

The notion of controllability is concerned with the following question—Given a linear time-invariant (LTI) system $\Sigma : \dot{x} = Ax + Bu$ with initial state x_0, initial time t_0, final state $x_f \in \mathbb{R}^n$, and final time $T \in \mathbb{R}$ (with $t < T$), does there exist any piecewise continuous control input $u_{[t_0,T]}$ that steers (x_0,t_0) to (x_f,T)?

If the answer to the above question is in the affirmative for *any* $x_0, x_f \in \mathbb{R}^n$, $t,T \in \mathbb{R}$, $t < T$, the system $\Sigma : \dot{x} = Ax + Bu$ is said to be *completely controllable* on $[t,T]$.

If there exists a u piecewise continuous such that $x(T,t_0,x_0,u) = x_f, \forall x_0, x_f, t_0, T > t_0$, then the system is (completely) controllable.

Definition 1.8 (**Complete Controllability on** $[t_0,t_1]$). *The system representation* D *is* ***(completely) controllable on*** $[t_0,t_1]$ *if, for each* $x_0,x_1 \in \Sigma$*, there exists some* $u_{[t_0,t_1]} \in \mathcal{U}$ *that steers* x_0 *at* t_0 *to* x_1 *at* t_1*.*

Definition 1.9 (**Reachable Set**). *Given a system* Σ*, the* ***reachable set*** *of* Σ *from* (x_0,t_0) *at time* $T > t_0$ *is defined by:*

$$R(T,t_0,x_0) = \{\tilde{x} \in \mathbb{R}^n | \exists \text{ piecewise continuous } u_{[t_0,T]} \ni x(T,t_0,x_0,u) = \tilde{x}\}.$$

Proposition 1.10. $R(T,t_0,0)$ *is a subspace of* $\mathbb{R}^n$*.*

Proof. It suffices to show that $\mathbb{R}(T, t_0, 0)$ is invariant under linear combination. Let $x_f^1, x_f^2 \in \mathbb{R}(T, t_0, 0)$. Then there exist inputs $u^1_{[t_0,T]}, u^2_{[t_0,T]}$ that steer the system from (x_0, t_0) to (x_f^1, T) and to (x_f^2, T), respectively. In other words, for $k = 1, 2$, we have:

$$x_f^k = e^{A(t-t_0)}0 + \int_{t_0}^{T} e^{A(t-\tau)} B u_k(\tau) d\tau.$$

Thus, we have:

$$\begin{aligned} x_f^k &= e^{A(t-t_0)}0 + \int_{t_0}^{T} e^{A(t-\tau)} B u_i(\tau) d\tau \\ &= \int_{t_0}^{T} e^{A(t-\tau)} B u_k(\tau) d\tau. \end{aligned}$$

We need to show that $\alpha_1 x_f^1 + \alpha_2 x_f^2 \in \mathbb{R}(T, t_0, 0)$. Plugging in, this is true because integration is linear, and a linear combination of piecewise continuous functions is still piecewise continuous:

$$\alpha_1 x_f^1 + \alpha_2 x_f^2 = \int_{t_0}^{T} e^{A(t-\tau)} B\big(\alpha_1 u_1(\tau) + \alpha_2 u_2(\tau)\big) d\tau.$$

■

Remark. This is a powerful result, since it considerably reduces the number of possibilities for what R can be. For instance, if $n = 3$, then $R(T, t_0, 0)$ is a subspace of $\mathbb{R}^3$, so $R(T, t_0, 0)$ is some rotation of a plane, some rotation fo a line, $\{0\}$, or $\mathbb{R}^3$.

Example. If $x_0 \in R(T, t_0, 0)$, it is not necessarily true that $R(T, t_0, x_0) = R(T, t_0, 0)$. For instance, if $B = 0$, the system is completely unaffected by any choice of input, in which case:

$$\begin{aligned} R(T, t_0, 0) &= \{0\} \\ R(T, t_0, x_0) &= \{e^{A(T-t_0)} x_0\}, \end{aligned}$$

and $R(T, t_0, 0)$ and $R(T, t_0, x_0)$ turn out to be distinct for any $x_0 \neq 0$. However, this statement is true if the system is completely controllable, in which case:

$$R(T, t_0, x_0) = \mathbb{R}^n,$$

for any $x_0 \in \mathbb{R}^n$, including 0.

Lemma 1.11. $R(T, t_0, x_0) = e^{A(T-t_0)} x_0 + R(T, t_0, 0)$. *Note that $R(T, t_0, x_0)$ is not a subspace in general, since it need not include the origin.*

Proof. Follows from linearity. ■

Theorem 1.12. *Consider the LTI system $\Sigma : \dot{x} = Ax + Bu$ with $x \in \mathbb{R}^n, u \in \mathbb{R}^{n_i}$, and fix arbitrary x_0 and t_0. Then the following are equivalent:*

1. *Σ is completely controllable.*
2. *$R(T, t_0, x_0) = \mathbb{R}^n$ for* all *$T > t_0$.*
3. *$R(T, t_0, x_0) = \mathbb{R}^n$ for* some *$T > t_0$.*
4. *$R\left(\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\right) = \mathbb{R}^n$.*
5. *$\operatorname{rank}\left(\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}\right) = n$.*

Proof. We will verify various parts of the above proof for the single-input case, i.e. for the case $\dot{x} = Ax + Bu, u \in \mathbb{R}$.

Observe that (3) naturally follows from (2), while (4) and (5) are equivalent by definition of range space and rank. What remains of the theorem follows from the following claim:

$$\text{Claim : } R(T, t_0, 0) = \text{span}\{b, Ab, \ldots, A^{n-1}b\}.$$

The claim holds because

$$\begin{aligned} x_f = x(T) &= \int_{t_0}^{T} e^{A(T-\tau)} b u(\tau) d\tau \\ &= \int_{t_0}^{T} \sum_{i=0}^{\infty} \alpha_i (T-\tau) A^i b\, u(\tau) d\tau \qquad \text{(Taylor Series Expansion)} \\ &= \sum_{i=0}^{\infty} A^i b \underbrace{\left(\int_{t_0}^{T} \alpha_i (T-\tau) u(\tau) d\tau \right)}_{\equiv \beta_i(T)} \\ &= \sum_{i=0}^{\infty} \beta_i(T) A^i b \\ &= \sum_{i=0}^{n-1} \gamma_i(T) A^i b \qquad \text{(Cayley-Hamilton Theorem)} \\ &\in \text{span}\{b, Ab, \ldots, A^{n-1}b\} \end{aligned}$$

One can also show the other direction (omitted in class). ■

Lemma 1.13 (PBH Test for Controllability). *$\Sigma : \dot{x} = Ax + Bu$ is completely controllable if and only if:*

$$\operatorname{rank}\left(\begin{bmatrix} \lambda I - A & B \end{bmatrix}\right) = n$$

for each $\lambda \in \mathbb{R}$.

Example. Let $\dot{x} = Ax + Bu$, with:

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

1. Is the system controllable? Answer: no.
2. Compute $R(T, t_0, 0)$.
3. Given $x_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, x_f = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$, does there exist a piecewise continuous control $u_{[t_0,T]}$ such that $x(T) = x_f$?
4. Given $x_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, x_f = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, does there exist a piecewise continuous control $u_{[t_0,T]}$ such that $x(T) = x_f$?
5. Given $x_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, x_f = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, does there exist a piecewise continuous control $u_{[t_0,T]}$ such that $x(T) = x_f$?

Solution :

1. No, since, by the PBH test, $\text{rank}\big(\begin{bmatrix} -3I - A & B \end{bmatrix} \big) = 2 < 3$.
2. Observe that:

$$R(T, t_0, 0) = R\big(\begin{bmatrix} b & Ab & A^2 b \end{bmatrix} \big) = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \right)$$

3. Yes, since $x_f \in R(T, t_0, 0)$.
4. No, since $x_f \notin R(T, t_0, 0)$.
5. We must check whether or not $x_f - e^{A(T-t_0)} \in R(T, t_0, 0)$ (Not done in class yet).

January 29, 2019

To begin with, consider the table below, which summarizes conditions for different notions of stability as they apply to linear systems. We indicate necessary conditions with **(n.)** and sufficient conditions with **(s.)**; all remaining, unmarked conditions are necessary and sufficient.

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Sometimes, it is desirable to have undesirable systems, to improve response time (e.g. for fighter jets). However, such systems should be controllable; this inspires the need for the concept of stabilizability.

Recall that an LTI system is stabilizable, by definition, if each of its poles in the closed right complex plane ($\overline{\mathbb{C}^+}$) can be relocated to the left half complex plane ($\mathbb{C}^-$) via state feedback, i.e. if each of its unstable modes is controllable. Equivalently, an LTI system is stabilizable if and only if each of its uncontrollable modes is already stable. Mathematically, if the LTI system $\Sigma : \dot{x} = Ax + Bu$, where $x \in \mathbb{R}^n, u \in \mathbb{R}^{n_i}$, is stable, then there exists some state feedback

Property	Eigenvalue test	Intuition
1. Stable in the sense of Lyapunov	$\text{Re}\{\lambda_i\} \leq 0, \forall i$ (**n.**)	Solution stays in neighborhood of x_{eq}
2. Asymptotic stability	$\text{Re}\{\lambda_i\} < 0, \forall i$	$x(t, x_0) \to x_{eq}$ as $t \to \infty$
3. Exponential stability	Same as asymptotic stability	Same as asymptotic stability
4. Unstable	$\exists i, \text{Re}\{\lambda_i\} > 0$ (**s.**)	$x(t, x_0) \to \infty$ for some x_0 sufficiently close to x_{eq}

Table 1.1: Stability for linear systems.

gain such that, if the state feedback $u = -Kx$ is applied to the system, the resulting closed loop system:

$$\Sigma_{SF} : \dot{x} = Ax + Bu$$

has poles that all lie in $\mathbb{C}^-$, i.e. $\sigma(A - BK) \in \mathbb{C}^-$.

In general, if the system is controllable, there exist two common methods for selecting the state feedback gain K:

1. **Pole Placement:** Fix a choice of closed-loop poles $\{p_1, \cdots, p_n\}$, and design the state feedback gain matrix K to place the closed-loop poles there, i.e.:

$$\sigma(A - BK) = \{p_1, \cdots, p_n\}.$$

2. **Linear Quadratic Regulator:** Fix performance parameters Q, R, where Q is positive semidefinite ($Q \geq 0$) and R is positive definite ($R > 0$), and design the state feedback $K(T)$ such that the cost:

$$J\big(u_{[0,T]}\big) = \int_0^T x^T Q x + u^T R u \, du$$

is minimized, subject to the dynamics constraints:

$$\dot{x} = Ax + Bu, \quad u = -K(t) \cdot x$$

When $T \to \infty$, the state feedback $K(t)$ becomes time-invariant.

Next, we wish to examine differences between linear and non-linear systems. First, we wish to ask ourselves why non-linear systems are worthy of our attention, when they may sometimes simply be linearized and controlled using linear system theory. The following list addresses the necessity of studying on-linear systems:

1. Linear control works only locally, if at all, on most non-linear systems.

2. Nonlinear control may allow the system to be stabilized with greater efficiency, speed, and/or accuracy.

3. Nonlinear systems may have properties or exhibit behaviors not easily described using linear system theory (e.g. multiple isolated equilibria, bifurcations, limit cycles).

4. Nonlinear systems can yield more interpretable results.

The following table provides a brief summary of the main differences between linear and nonlinear systems.

Property	LTI system ($\dot{x} = Ax + Bu$)	Nonlinear system ($\dot{x} = f(x, u)$)
Equilibria	1. Equilibria are singletons (0) or connected ($N(A)$)	1. No general form
	2. $\text{Re}\{\lambda_i\} < 0 \Rightarrow$ Global stability 3. Very fragile with respect to elements of A	2. May have the following: (a) Bounded regions of attraction (b) Limit cycles (Isolated periodic order) (c) Multiple isolated equilibria
	4. System stability is independent of the choice of input u	4. System stability may depend on the choice of input u
Solutions	1. Solution (analytic) uniquely exists for each $t \geq 0$: ($x(t) = e^{At}x_0$)	1. Solution: (a) May not always exist (b) May exist non-uniquely (e.g. Bifurcations) (c) May exist uniquely, but only in some finite time range (e.g. Finite escape time)
	2. Linear dependence on initial conditions (x_0, t_0)	2. Extreme Sensitivity to initial conditions (x_0, t_0)

Table 1.2: Properties of linear and nonlinear systems.

Below, we provide examples that illustrate each of the eccentricities of nonlinear systems listed above.

Example (**Finite Escape Time**). Consider the nonlinear system given by:

$$\Sigma : \ \dot{x} = -x + x^2, \qquad x(0) = x_0.$$

The solution to the system is given by:

$$x(t) = \frac{x_0 e^{-t}}{1 - x_0(1 - e^{-t})}$$

If $x_0 < 1$, the above solution is valid for each time t. If $x_0 = 1$, the original differential equation gives the solution $x(t) = 1$. If $x_0 > 1$, then the solution of the above system holds only in the range $\left(0, \ln\left(\frac{x_0}{x_0 - 1}\right)\right)$; after that point, the solution diverges to infinity.

Example (**Stable, Unstable, and Semi-Stable Equilibria**). Consider the three following systems, each expressed in polar coordinates (with $r \in [0, \infty), \pi \in [0, 2\pi)$) with arbitrary initial conditions:

$$\Sigma_1 : \dot{r} = -r(r^2 - 1), \qquad \dot{\theta} = 1,$$
$$\Sigma_2 : \dot{r} = r(r^2 - 1), \qquad \dot{\theta} = 1,$$
$$\Sigma_3 : \dot{r} = r(r^2 - 1)^2, \qquad \dot{\theta} = 1.$$

Observe that $\Sigma_1, \Sigma_2, \Sigma_3$ each have equlibria at $r = 0, 1$. In particular, regarding the equilibrium point $r = 1$:

1. $r = 1$ is a stable equilibrium point for Σ_1, since $\dot{r} > 1$ when $r < 0$, and $\dot{r} < 0$ when $r > 1$.
2. $r = 1$ is an unstable equilibrium point for Σ_2, since $\dot{r} < 0$ when $r < 1$, and $\dot{r} > 0$ when $r > 1$.
3. $r = 1$ is an semi-stable equilibrium point for Σ_3, since $\dot{r} > 0$ for each r.

Example (**System Stability may depend on input**). Consider the system:

$$\Sigma : \dot{x} = xu,$$

with $\|u\|_2 \leq 1$. If $u \in [-1, 0]$, the system is stable; if $u \in [-1, 0]$, the system is unstable.

Example (**Existence and Uniqueness of Solutions**). Consider the systems:

$$\Sigma_1 : \dot{x} = -\text{sgn}(x), \qquad x(0) = 0.$$
$$\Sigma_2 : \dot{x} = -3x^{2/3}, \qquad x(0) = 0,$$
$$\Sigma_3 : \dot{x} = -x + x^2, \qquad x(0) = 0.$$

We have the following analysis:

- Σ_1 has no solution in C^1.
- Σ_2 has the following two solutions:

$$x_1(t) = t^3,$$
$$x_2(t) = 0$$

- Σ_3 has the following unique solution:

$$x(t) = \frac{x_0 e^{-t}}{1 - x_0(1 - e^{-t})},$$

  which holds only in the time frame $\left(0, \ln\left(\frac{x_0}{x_0 - 1}\right)\right)$

Below, we examine the phenomenon of bifurcations. Systems of physical interest often have parameters which appear in the defining systems of equations:

$$\dot{x}_1 = f_{1,\mu}(x_1, x_2),$$
$$\dot{x}_2 = f_{2,\mu}(x_1, x_2)$$

When a non-linear system $\dot{x} = f(x, u, t)$ has a singular Jacobian at the point of linearization, several branches of equilibria can come together. As these parameters μ are varied, changes occur in the qualitative structure of the solutions, denoted as $x^\star(\mu)$. We have the following result, derived from the Implicit Function Theorem:

Theorem 1.14 (**Implicit Function Theorem**). *Consider a parameterized non-linear system $\dot{x} = f_\mu(x, t)$. If the Jacobian linearization about $x^\star(\mu)$, denoted by:*

$$D_x f_\mu(x^\star(\mu)) \equiv \frac{\partial f_\mu}{\partial x}\big(x^\star(\mu)\big),$$

does not have a zero eigenvalue, the solution $x^\star(\mu)$ is a smooth function of μ.

However, when $D_x f_\mu\big|_{x^\star(\mu)}$ has a zero eigenvalue, several branches of equilibria may come together. When this occurs, $x^\star(\mu)$ is called a **bifurcation point**.

Example (**Bifurcation Point**). Consider the scalar system given by:

$$\dot{x} = \mu x - x^3,$$

where $\mu \in \mathbb{R}$. If $\mu \leq 0$, the system has only one equilibrium, $x_{eq} = 0$. However, when $\mu > 0$, there are three possible equilibria—$x_{eq} = 0, \pm\sqrt{\mu}$. This is illustrated by the figure below.

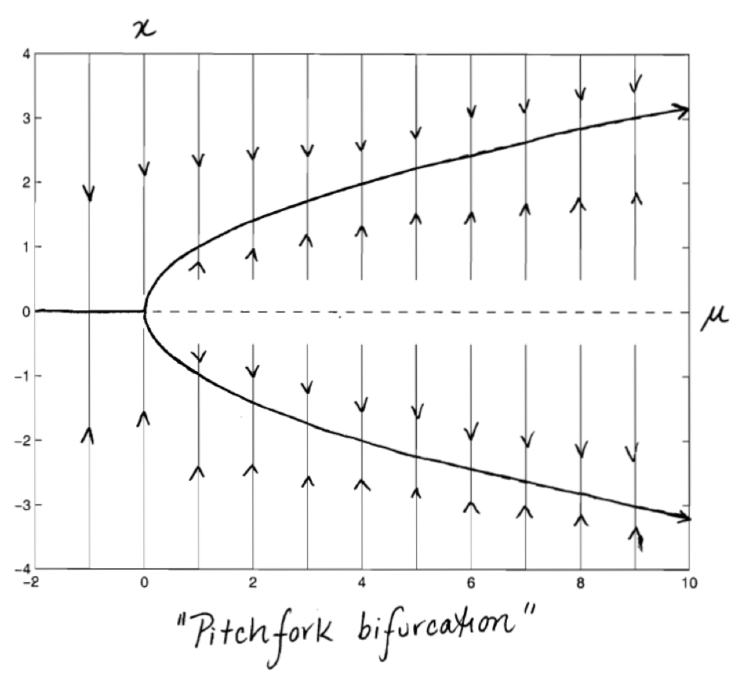

Figure 1.1: Pitchfork bifurcation

1.2 Mathematical Preliminaries:

January 31, 2019

Next, we will review some basic definitions and concepts from real analysis. The motivation for these concepts is as follows— Consider the nonlinear system:

$$\Sigma : \dot{x} = f(t, x, u), x(0) = x_0, t \geq 0$$

. We wish to examine the following properties of the system:

- *(Existence).* When does Σ have *at least one* solution?
- *(Uniqueness).* When does Σ have *exactly one* solution?
- When does Σ have one solution defined on $t \in [0, \infty)$?
- When does Σ have one solution defined on $t \in [0, \infty)$ that depends continuously on x_0.

Definition 1.15 (**Extended Real Line**)**.** *Define the **extended real line** $\mathbb{R}_e$ (or $\overline{\mathbb{R}}$) as the union of $\mathbb{R}$ and $\{\pm\infty\}$, i.e.:*

$$\bar{\mathbb{R}} = \mathbb{R}_e = \mathbb{R} \cup \{\pm\infty\}$$

Definition 1.16 (**Supremum**)**.** *Let $S \subset \mathbb{R}$. An element $a^\star \in \mathbb{R}_e$ is called the **supremum**, or **least upper bound**, of S if:*

- $s \leq a^\star, \forall s \in S$
- *If $b \in \mathbb{R}_e$ satisfying $s \leq b, \forall s \in S$, then $a^\star \leq b$.*

Definition 1.17 (**Infimum (Greatest Upper Bound)**)**.** . *Let $S \subset \mathbb{R}$. An element $a_\star \in \mathbb{R}_e$ is called the **infimum**, or **greatest lower bound**, of S if:*

- $s \geq a_\star, \forall s \in S$
- *If $c \in \mathbb{R}_e$ satisfying $s \geq c, \forall s \in S$, then $a_\star \geq c$.*

Proposition 1.18. *The following is always true of any subset S of $\mathbb{R}$.*

1. *The supremum of S always uniquely exists.*
2. *If the maximum of S exists, it must be equal to its supremum.*
3. *The infimum of S always uniquely exists.*
4. *If the minimum of S exists, it must be equal to its infimum.*

Example. A few examples of suprema are as follows:

- $S = (0, 1)$. $\sup S = 1$, but $\max S$ does not exist.
- $S = (0, 1]$. $\sup S = 1 = \max S$.
- S $= \{x \geq 0 \mid x \in \mathbb{R}\}$. $\sup S = \infty$.

Definition 1.19 (**Supremum of a Function**). *Suppose $S \subset \mathbb{R}, F : S \to \mathbb{R}$, then*

$$\sup_{x \in S} f(x) := \sup\{f(x) \mid x \in S\}$$

Definition 1.20 (**Norms**). *Let V be a vector space over the reals $\mathbb{R}$. Then $\|\cdot\| : V \to \mathbb{R}$ is a **norm over** V if:*

- $\forall x, \|x\| \geq 0$ *and* $\|x\| = 0 \iff x = 0$
- $\forall x, \alpha \in \mathbb{R}, \|\alpha x\| = |\alpha| \cdot \|x\|$
- $\forall x, y, \|x + y\| \leq \|x\| + \|y\|$

Definition 1.21 (**Normed Space**). *$(V, \|\cdot\|)$ is called a **normed space** if V is a vector space over $\mathbb{R}$ and $\|\cdot\| : V \to \mathbb{R}$ is a norm.*

Example. Let $x \in \mathbb{R}^n$, then:

- $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2} = \sqrt{x^T x}$
- $\|x\|_1 = \sum_{i=1}^n |x_i|$
- $\|x\|_\infty = \max_i |x_i|$

Now let $V = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous}\} = C[a, b]$

- $\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|$

Definition 1.22 (**Induced norms**). *Let $A : V \to V$ be a linear operator and $(V, |\cdot|)$ be a normed space. The induced norm of A is:*

$$|A|_i := \sup_{x \neq 0} \frac{|Ax|}{|x|} = \sup_{|x|=1} \|Ax\|$$

Proposition 1.23. $\|Ax\| \leq \|A\|_i \|x\|, \forall x \in V$

Example. $(V, \|\cdot\|) = (\mathbb{R}^n, \|\cdot\|_2$, then $\|A\|_i = \sqrt{\lambda_{max}(A^T A)}$.

Proof. $\|Ax\| = \sqrt{(Ax)^T Ax} = \sqrt{x^T A^T Ax} \ldots$ ■

Definition 1.24 (**Equivalence of norms**). *Two norms $\|\cdot\|_a, \|\cdot\|_b$ on V are **equivalent** if $\exists k_1, k_2 > 0$ such that, for each $x \in V$:*

$$k_1 \|x\|_a \leq \|x\|_b \leq k_2 \|x\|_a \ .$$

Theorem 1.25. *A vector space V is finite dimensional if and only if all norms on V are equivalent.*

Definition 1.26. *Open sets and open balls. Let $x_0 \in V, (V, \|\cdot\|)$ be a normed space, and let $a > 0$.*

1. *The open ball of radius a centered at x_0 is defined as:*

$$B_a(x_0) := \{x \in V \mid \|x - x_0\| < a\}$$

2. *A set $S \subset V$ is* open *if $\forall s_0 \in S, \exists \epsilon > 0$, s.t. $B_\epsilon(x_0) \subset S$.*

Example. A few examples of open balls include the following:

- $(\mathbb{R}^2, \|\cdot\|_2), B_1(0)$ is the open unit disc.
- $(\mathbb{R}^2, \|\cdot\|_1), B_1(0)$ is a diamond.
- $(\mathbb{R}^2, \|\cdot\|_\infty), B_1(0)$ is a square.

Remark. By convention, the empty set $\emptyset$ is open.

Example. A few examples of open sets.

- $S = (0, 1) \subset (\mathbb{R}, |\cdot|)$ is open
- $S = [0, 1) \subset (\mathbb{R}, |\cdot|)$ is *not* open ($s_0 = 0$ is a counterexample). In fact it is not closed either.

Definition 1.27 (Closed Set). *A set S is closed if its complement S^c is open.*

Proposition 1.28. *The following facts hold for arbitrary open and closed sets:*

1. *The union of a collection of open sets is open.*
2. *The union of a finite collection of closed sets is closed.*
3. *The intersection of a collection of closed sets is closed.*
4. *The intersection of a finite collection of open sets is open.*

Definition 1.29 (Convergence of a Sequence). *Let $(x_k, k \geq 1)$ denote a sequence of vectors in some normed space $(V, |\cdot|)$. We say that $(x_k, k \geq 1)$* ***converges to a point*** *$\overline{x}$ if, for each $\epsilon > 0$, there exists some $N(\epsilon) \in \mathbb{N}$, such that:*

$$\|x_k - \overline{x}\| < \epsilon,$$

for each $n \geq N(\epsilon)$.

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Note (**Notation**). If $(a_n, n \geq 1)$ is a sequence that converges to a, we write:

1. $\lim_{n\to\infty} a_n = a$, or
2. $(a_n) \to a$ as $n \to \infty$.

Example. Consider the following sequences in $(\mathbb{R}, |\cdot|)$:

- If $(x_n) = 1/n$, then $\lim_{n\to\infty} x_n = 0$.
- If $(x_n) = (1 + 1/n)^n$, then $\lim_{n\to\infty} x_n = e$.

Next, we will discuss complete spaces and contraction mappings.

Definition 1.30 (**Cauchy Sequence**). *A sequence (x_n) is* ***Cauchy*** *if, for each $\epsilon > 0$, there exists some $N > 0$ such that*

$$|x_n - x_m| < \epsilon$$

for each $n, m \geq N$.

Theorem 1.31. *If $(x_n, n \geq 1)$ is a convergent sequence in X, then (x_n) is a Cauchy sequence.*

Proof. Let $\epsilon > 0$, and suppose $x_n \to \bar{x}$ for some $\bar{x} \in X$. Then there exists some $N \in \mathbb{N}$ such that for each $n \geq N$:,

$$|x_n - \bar{x}| < \frac{1}{2}\epsilon.$$

Now, fix $m, n \geq N$. Then:

$$\begin{aligned} |x_n - x_m| &= |x_n - \bar{x} + \bar{x} - x_m| \\ &\leq |x_n - \bar{x}| + |x_m - \bar{x}| < \epsilon \end{aligned}$$

Thus, by definition, (x_n) is a Cauchy sequence. ■

Definition 1.32 (**Complete Space**). *A vector space $(V, |\cdot|)$ is* complete *if every Cauchy sequence in V converges to a point in V.*

We state the following facts without proof.

Theorem 1.33.

1. *Every finite dimensional vector space is complete.*
2. *$(C[a,b], |\cdot|_\infty)$ is complete.*

Definition 1.34 (**Banach Space**). *A complete normed space is called a* ***Banach space****.*

Definition 1.35 (**Continuous Function**). *Let* $(V, \|\cdot\|_V)$ *and* $(W, \|\cdot\|_W)$ *be normed spaces.*

1. *A function* $f : V \to W$ *a* ***continuous at*** $x_0 \in V$ *if, for each* $\epsilon > 0$*, there exists some* $\delta > 0$ *such that whenever* $\|x - x_0\|_V < \delta$*, we have:*

$$\|f(x) - f(x_0)\|_W < \epsilon.$$

2. *A function* $f : V \to W$ *is* ***continuous on*** V *if it is continuous at all points* $x_0 \in V$*.*

Below, we will introduce the contraction mapping theorem. First, consider the following conditions

Definition 1.36 (**Lipschitz Continuity, Contraction**). *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed vector spaces.*

1. *The function* $f : X \to Y$ *is said to be* ***(locally) Lipschitz continuous at*** $x_0 \in X$ ***with Lipschitz constant*** κ*, if there exists some* $\kappa > 0, r > 0$ *such that:*

$$\|f(x) - f(y)\|_Y \leq \kappa \cdot \|x - y\|_X$$

whenever $x, y \in B_r(x_0)$*. This inequality is called the* ***Lipschitz condition****, and* $\kappa > 0$ *is called the* ***Lipschitz constant****.*

2. *The function* $f : X \to Y$ *is said to be* ***globally Lipschitz continuous at*** $x_0 \in X$ ***with Lipschitz constant*** κ*, if there exists some* $\kappa > 0$ *such that:*

$$\|f(x) - f(y)\|_Y \leq \kappa \cdot \|x - y\|_X$$

for each $x, y \in X$*.*

3. *The function* f *is called a* ***contraction*** *if it is Lipschitz continuous with Lipschitz constant* $c \in [0, 1)$*, i.e. if there exists some* $c \in [0, 1)$ *such that, for each* $x, y \in X$*:*

$$\|f(x) - f(y)\|_Y \leq c \cdot \|x - y\|_X$$

.

Definition 1.37 (**Fixed Point**). *Let* $(X, \|\cdot\|_X)$ *be a normed vector space, and let* $f : X \to X$ *be given. We say that* $x^* \in X$ *is a* ***fixed point of*** f *if* $f(x^*) = x^*$*.*

Theorem 1.38 (**Contraction Mapping Theorem**). *Let* $(X, |\cdot|)$ *be a Banach space, and let* $f : X \to X$ *be a contraction with Lipschitz constant* c*. Then the following statements hold:*

1. f *has a unique fixed point, i.e. there exists a unique* $x^* \in X$ *such that* $f(x^*) = x^*$*.*

2. *Fix* $x_0 \in X$*, and define* $x_n = f(x_{n-1})$*, for each* $n \in \mathbb{N}$*. Then* (x_n) *is Cauchy, and converges to* x^**.*

3. *The rate of convergence of* (x_n) *to* $x^\star$ *decreases at least as fast as* $O(c^n)$*.*

Example. $(\mathbb{R}^2, |\cdot|_2)$ and seek to solve $Ax = b$. Let $P(x) = x + (Ax - b)$.

$$\begin{aligned} |P(x) - P(y)|_2 &= |x + Ax - b - y - Ay + b|_2 \\ &= |(A + I)(x - y)|_2 \\ &\leq \|A + I\|_i \cdot |x - y|_2 \end{aligned}$$

so P is a contraction if $\|A + I\|_i = \sigma_{max}(A + I) < 1$.

Note that:

$$\begin{aligned} P(x^*) = x^* &\iff x^* = x^* + (Ax^* - b) \\ &\iff Ax^* = b \end{aligned}$$

We state the following theorem without proof, although the proof is not difficult.

Theorem 1.39. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be given.*

1. *If f is Lipshitz continuous at some $x_0 \in \mathbb{R}^n$, then it is continuous at x_0.*
2. *If f is differentiable, and there exists $x_0 \in X, r > 0, L < \infty$ such that:*

$$\left\| \frac{\partial h}{\partial x}(x) \right\| < L$$

for each $x \in B_r(x_0)$, then f is locally Lipschitz at x_0.

Example. The saturation function $h(x)$, given by:

$$h(x) = \begin{cases} -1, & x < 1, \\ |x|, & x \in [-1, 1], \\ 1, & x > 1. \end{cases}$$

is Lipschitz continuous with Lipschitz constant $L = 1$ (in fact, one can take any $L \geq 1$ to be the Lipschitz constant for h).

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Definition 1.40 (Solution to an ODE). *Given an ODE:*

$$\dot{x} = f(x, t), \qquad x(t_0) = x_0, \tag{1.1}$$

where $t \geq t_0, x_0 \in \mathbb{R}^n$, we say that $\phi(t)$ is a solution of Σ on $[t_0, t_1]$ if the following conditions hold:

1. *$\phi(t)$ is differentiable on $[t_0, t_1]$,*
2. *$\phi(t)$ satisfies $\dot{\phi}(t) = f\big(\phi(t), t\big), \qquad \forall t \in [t_0, t_1]$.*

3. $\phi(t_0) = x_0$,

or, more generally, if the following conditions hold:

1. $\phi(t)$ *is integrable on* $[a, b]$,

2. $\phi(t)$ *satisfies:*

$$\phi(t) = x_0 + \int_{t_0}^{t} f\big(\tau, f(\tau)\big)\, d\tau$$

for each $t \in [a, b]$.

Theorem 1.41 (Local Existence and Uniqueness of Solutions of an ODE). *Consider the differential equation:*

$$\dot{x} = f(x, t), \qquad x(t_0) = x_0$$

where $f(x, t) : \mathbb{R}^n \times R_+ \to \mathbb{R}^n$ *is piecewise continuous in* t *and Lipschitz continuous at each* x, *i.e. there exists some* $T > t_0, r > 0, L > 0$ *such that:*

$$\|f(t, x) - f(t, y)\| \leq L \cdot \|x(t) - y(t)\|$$

for each $x, y \in B_r(x_0)$ *and* $t \in [t_0, T]$. *Then there exists a unique function of time* $\phi(\cdot) : [t_0, t_0 + \delta] \to \mathbb{R}^n$ *that is continuously differentiable almost everywhere, and is the solution to the given ODE, i.e. it satisfies:*

$$\phi(t_0) = x_0$$
$$\dot{\phi}(t, 0) = f(\phi(t), t),$$

for each $t \in [t_0, t_0 + \delta] \backslash D$, *where* D *is the set of discontinuity points of* f *as a function of* t.

Proof. (**Sketch**) We wish to apply the Contraction Mapping Theorem. Consider the family of continuous functions $C_n[t_0, t_0 + \delta]$, as defined below:

$$C_n[t_0, t_0 + \delta] = \{\phi(\cdot) : [t_0, t_0 + \delta] \in \mathbb{R}^n | \phi \text{ is continuous}\},$$

for any $\delta > 0$. For each $\phi(\cdot) \in C_n[t_0, t_0 + \delta]$, define the infinity norm of ϕ as:

$$\|\phi(\cdot)\|_\infty \equiv \sup_{t \in [t_0, t_0 + \delta]} \|\phi(t)\|.$$

We are now ready to define our contraction. Consider $P : C_n[t_0, t_0 + \delta] \to C_n[t_0, t_0 + \delta]$, defined by:

$$(P \circ \phi)(t) = x_0 + \int_{t_0}^{t} f\big(\tau, \phi(\tau)\big)\, d\tau, \qquad \forall t \in [t_0, t_0 + \delta]$$

for each $\phi(\cdot) \in C_n[t_0, t_0 + \delta]$. Thus, for any $x(\cdot), y(\cdot) \in C_n[t_0, t_0 + \delta]$, we have:

$$\begin{aligned}
\|(P \circ x)(t) - (P \circ y)(t)\|_\infty &= \sup_{t \in [0,\delta]} \left\| \int_{t_0}^t f\big(x(\tau), \tau\big) - f\big(y(\tau), \tau\big)\, d\tau \right\| \\
&\leq \sup_{t \in [0,\delta]} \int_{t_0}^t \big\| f\big(x(\tau), \tau\big) - f\big(y(\tau), \tau\big) \big\|\, d\tau \leq \sup_{t \in [0,\delta]} L \cdot \int_{t_0}^t \|x(\tau) - y(\tau)\|\, d\tau \\
&\leq \sup_{t \in [0,\delta]} L \cdot \int_{t_0}^t \|(x - y)(\cdot)\|_\infty\, d\tau = \sup_{t \in [0,\delta]} L \cdot \|x - y\|_\infty \cdot (t - t_0) \\
&\leq L\delta \cdot \|x - y\|_\infty
\end{aligned}$$

Take $\delta < 1/L$. Then P is a contraction, so it has a unique fixed point, at which:

$$\begin{aligned}
x^\star(t) &= x_0 + \int_{t_0}^t f\big(\tau, x^\star(\tau)\big)\, d\tau \\
\Leftrightarrow \dot{x}^\star(t) &= f\big(x^\star(t), t\big).
\end{aligned}$$

■

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Below, we present a different version of Theorem 1.41.

Corollary 1.42. *Consider the differential equation* (1.1) *(reproduced below):*

$$\dot{x} = f(x, t), \qquad x(t_0) = x_0,$$

where $f(t, x)$ is piecewise continuous in t, and there exists some $T > t_0$, $r > 0$, $L > 0$ such that:

$$\left\| \frac{\partial f}{\partial x}(t, x) \right\| \leq L,$$

for each $x \in B_r(x_0)$ and $t \in [t_0, T]$. Then there exists some $\delta > 0$ such that a unique solution to (1.1) *exists on* $[t_0, t_0 + \delta]$.

Remark. Notice that the conclusion of this corollary is identical to that of the above theorem; however, here the assumptions on f are stronger. Whereas previously, we only required f to be locally Lipschitz continuous in x about x_0 and piecewise continuous in t, here we require the existence and boundedness of partial derivatives of f in some neighborhood centered at x_0.

Below, we present a global version of Theorem 1.41.

Theorem 1.43 (Global Existence and Uniqueness of Solutions of an ODE). *Consider the differential equation* (1.1) *(reproduced below):*

$$\dot{x} = f(x, t), \qquad x(t_0) = x_0,$$

If $f(t, x)$ is piecewise continuous in t, and for each $T \in [t_0, \infty)$, there exists a finite constant L_T such that:

$$\|f(t, x) - f(t, y)\| \leq L_T \cdot \|x - y\|$$

for each $x, y \in \mathbb{R}^n$ and $t \in [t_0, T]$. Then the ODE given by (1.1) *has exactly one solution on* $[t_0, T]$.

Consider the examples below.

Example.

1. Consider the LTi system given by:

$$\dot{x} = Ax + Bu(t), \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

Here, $f(t, x) = Ax + Bu(t)$, so:

$$\|f(t, x_1) - f(t, x_2)\| = \|A(x_1 - x_2)\| \leq \|A\|_2 \cdot \|x_1 - x_2\|.$$

For this example, Theorem 1.43 implies global existence and uniqueness conditions hold when $\|A\|_i < \infty$, and $u(t)$ is piecewise continuous in t.

2. Consider the system given by:

$$\dot{x} = 1 + x^2, \qquad x(0) = 0.$$

Solving the ODE directly, we find that $x(t) = \tan(t)$ is a solution, but only in the time range $t \in [0, \pi/2)$. Indeed, for this example, Theorem 1.41 implies local existence and uniqueness of the solution, as is verified below, but implies nothing regarding the existence and uniqueness a possible global solution—For each $m \in \mathbb{R}^+$, when $\|x\| \leq m$, we have:

$$\left\| \frac{\partial f}{\partial x} \right\| < 2m.$$

Remark. (*Motivation for the Bellman-Gronwell-Inequality*) Consider the system:

$$\Sigma : \dot{y}(t) = \mu(t)\, y(t), \qquad y(a) = \lambda.$$

Equivalently, we have:

$$y(t) = \lambda + \int_0^t \mu(\tau)\, y(\tau)\, d\tau$$

Intuitively, we expect that the above relations continue to hold true if we replace the equalities ("$=$") with inequalities ("$\leq$"), i.e. we intuitively expect the following implication to be true:

$$y(t) \leq \lambda + \int_0^t \mu(\tau)\, y(\tau)\, d\tau,$$
$$\Rightarrow y(t) \leq \lambda \cdot \exp\left(\int_a^t u(\tau)\, d\tau \right).$$

In other words, we wish to verify the intuition that implicit inequalities on $y(t)$ can be rewritten into explicit ones. We formulate this rigorously below.

Theorem 1.44. *Let $\mu : [a,b] \to \overline{\mathbb{R}^+}$ and $\lambda \in \mathbb{R}$ be given. If $y : [a,b] \in \mathbb{R}$ is continuous, and:*

$$y(t) \leq \lambda + \int_0^t \mu(\tau)\, y(\tau)\, d\tau \tag{1.2}$$

then:

$$y(t) \leq \lambda \cdot \exp\left(\int_a^t \mu(\tau)\, d\tau\right)$$

Proof. Without loss of generality, suppose $t > t_0$. Define:

$$Z(t) = \lambda + \int_{t_0}^t \mu(\tau) y(\tau)\, d\tau,$$

then $u(t) \leq Z(t)$. In differential form:

$$\begin{aligned} \frac{d}{dt} Z(t) &= \mu(t) y(t) \\ Z(t_0) &= \lambda \end{aligned}$$

Multiplying both sides of (1.44) by the non-negative function:

$$\mu(t) e^{-\int_{t_0}^t \mu(\tau)\, d\tau} \geq 0$$

we find:

$$\begin{aligned} 0 &\geq \big[y(t) - Z(t)\big] \cdot \mu(t) e^{-\int_{t_0}^t \mu(\tau)\, d\tau} \\ &\geq \left(\frac{d}{dt} Z(t) - Z(t)\mu(t)\right) \cdot e^{-\int_{t_0}^t \mu(\tau)\, d\tau} \\ &= \frac{d}{dt}\left(Z(t) \cdot e^{-\int_{t_0}^t \mu(\tau)\, d\tau} - \lambda\right) \end{aligned}$$

Thus, the function $Z(t) \cdot e^{-\int_{t_0}^t \mu(\tau)\, d\tau} - \lambda$ is decreasing, and must at any time t be less than its value at t_0:

$$\begin{aligned} Z(t) \cdot e^{-\int_{t_0}^t \mu(\tau)\, d\tau} - \lambda &\leq Z(t_0) - \lambda = 0 \\ \Rightarrow y(t) \leq Z(t) &\leq \lambda \cdot e^{-\int_{t_0}^t \mu(\tau)\, d\tau} \end{aligned}$$

■

We continue to our final topic, that of whether the solutions to an ODE display continuous dependence on initial conditions.

Definition 1.45 (**Continuous Dependence on Initial Conditions**)**.** *Consider the differential equation* (1.1) *(reproduced below):*

$$\dot{x} = f(x,t), \qquad x(t_0) = x_0,$$

and the map:

$$\psi_T : (\mathbb{R}^n, \|\cdot\|) \to \big(C_n[t_0, T], \|\cdot\|_\infty\big)$$

defined by $\psi(x_0) = x(\cdot, t, x_0)$. *Then* ψ_T *is continuous at* x_0 *if, for each* $\epsilon > 0$, *there exists some* $\delta > 0$ *such that:*

$$\|x(t, t_0, x_0) - x(t, t_0, z_0)\| < \epsilon,$$

for each t, *whenever* $z_0 \in B_\delta(x_0)$.

Theorem 1.46 (**Global Dependence on Initial Conditions**). *Consider the ODE given by:*

$$\Sigma : \dot{x} = f(t, x),$$

where f *is globally Lipschitz continuous in* x, *and piecewise continuous in* t. *Fix* $T \in [t_0, \infty)$, *and suppose* $x(\cdot), z(\cdot)$ *satisfy:*

$$\begin{aligned} \dot{x} &= f(t, x), & x(t_0) &= x_0, \\ \dot{z} &= f(t, z), & z(t_0) &= z_0, \end{aligned}$$

respectively. Then, for each $\epsilon > 0$, *there exists some* $\delta > 0$ *such that:* $\sup_{t\in[t_0,T)} \|x(t) - z(t)\| < \epsilon$, *whenever* $\|x_0 - z_0\| < \delta$.

Remark. Essentially, ψ maps each state $x_0 \in \mathbb{R}^n$ to the trajectory the system would take, if the initial condition were x_0.

Proof. Rewrite the ODE, as satisfied by $x(\cdot)$ and $z(\cdot)$, as follows:

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} f\big(\tau, x(\tau)\big)\, d\tau, \\ z(t) &= z_0 + \int_{t_0}^{t} f\big(\tau, z(\tau)\big)\, d\tau. \end{aligned}$$

Applying the triangle inequality twice, followed by the Bellman-Gronwell inequlaity, we have:

$$\begin{aligned} \|x(\cdot) - y(\cdot)\| &\le \|x_0 - z_0\| + \int_{t_0}^{t} \|f\big(\tau, x(\tau)\big) - f\big(\tau, z(\tau)\big)\, d\tau \\ &\le \|x_0 + z_0\| + \int_{t_0}^{t} L_T \cdot \|x(\tau) - z(\tau)\|\, d\tau, \\ \Rightarrow \|x(t) - y(t)\| &\le \|x_0 - y_0\| \cdot e^{L_T(t-t_0)} \le e^{L_T(t-t_0)} \cdot \|x_0 - y_0\|. \end{aligned}$$

Taking $\delta \equiv e^{-L_T(t-t_0)} \cdot \epsilon$ completes the proof. ■

Remark. This result does not hold over infinite time intervals, e.g. $\dot{x} = L_T x$, for which a tight bound is reached.

Chapter 2

Nonlinear Systems

:

February 14, 2019

Before diving into nonlinear systems theory, we will review the following results from mathematical analysis.

Definition 2.1. *Let S be a subset of $\mathbb{R}^n$*

1. *S is called **open** if, for each $x_0 \in S_0$, there exists some $r > 0$ such that $B_r(x_0) \subset S$.*
2. *S is called **closed** if its complement in $\mathbb{R}^n$, i.e. $\mathbb{R}^n \backslash S$, is closed.*
3. *S is called bounded if there exists some $K \in \mathbb{R}^+$ such that, for each $x \in S$, we have $\|x\| \leq K$.*
4. *S is called **compact** if it is closed and bounded.*

Remark. In general, the above definition of compactness will not hold in infinite-dimensional spaces.

Lemma 2.2.

1. *If $V : \mathbb{R}^n \to \mathbb{R}$ is continuous, then for each $c \in \mathbb{R}$, the set $V^{-1}\big((-\infty, c]\big)$ is closed in $\mathbb{R}^n$.*
2. *If $h : \mathbb{R} \to \mathbb{R}$ is decreasing and bounded below, then there exists some unique $c \in \mathbb{R}$ such that:*
$$\lim_{x\to\infty} h(x) = c.$$
3. *If $V : \mathbb{R}^n \to \mathbb{R}$ is continuous on a compact set S, it is uniformly continuous on S.*
4. *If $V : \mathbb{R}^n \to \mathbb{R}$ is continuous, with $V(0) = 0$, $V(x) > 0$ for each $x \neq 0$, and $V(x_n) \to \infty$ for every unbounded sequence $(x_n) \subset \mathbb{R}^n$, then for each $c \geq 0$, the sublevel set: $L(c) \equiv V^{-1}\big((-\infty, c]\big)$ is compact.*

5. *If $V : \mathbb{R}^n \to \mathbb{R}$ is continuous, and K is a compact subset of $\mathbb{R}^n$, then V attains maxima and minima in K. (This is known as the Weierstrass extreme value theorem).*

Proof. For detailed proofs, the reader is referred to [9] or [1]. Below, we only prove the fourth lemma, namely—If $V : \mathbb{R}^n \to \mathbb{R}$ is continuous, with $V(0) = 0$, $V(x) > 0$ for each $x \neq 0$, and $V(x_n) \to \infty$ for every unbounded sequence $(x_n) \subset \mathbb{R}^n$, then for each $c \geq 0$, the sublevel set $L(c) \equiv V^{-1}\big((-\infty, c]\big)$ is compact.

Fix some $c > 0$. By the first lemma above, $V^{-1}\big((-\infty, c]\big)$ is closed. Since $V(x) \to \infty$ as $\|x\| \to \infty$, there exists some $R > 0$ such that for each x satisfying $\|x\| \geq R$, we have $V(x) > c$. The converse of this statement gives us:

$$V^{-1}\big((-\infty, c]\big) \subset B_R(0).$$

■

Theorem 2.3 (**[6], Theorem 3.3**). *Suppose $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is Lipschitz in x in some subset D of $\mathbb{R}^n$. Moreover, suppose there exists some compact subset W of D such that, for each $x_0 \in W$, the solutions to:*

$$\dot{x} = f(x, t), \qquad x(t_0) = x_0,$$

lie in W. Then this solution exists and is unique for each $t \geq t_0$.

Below, we begin our discussion of Lyapunov stability.

Definition 2.4 (**Stable in the Sense of Lyapunov**). *Let $\dot{x} = f(x)$ be a time-invariant system with equilibrium point x_e, and suppose there exists some $r > 0$ such that, if $x_0 \in B_r(x_e)$, the solution $x(t, x_0)$ uniquely exists for all $t \geq 0$. Then the equilibrium point x_e may be described as follows:*

1. *x_e is called **stable in the sense of Lyapunov** if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that:*
$$\|x(t, x_0) - x_e\| < \epsilon$$
whenever $\|x_0 - x_e\| < \delta$.

2. *x_e is called **unstable** if it is not stable.*

3. *x_e is called **asymptotically stable** if:*

   - *It is stable in the sense of Lyapunov.*
   - *There exists some $\eta > 0$ such that when $\|x_0 - x_e\| < \eta$, we have $\lim_{t\to\infty} \|x(t, x_0) - x_e\| = 0$.*

Remark. An alternative definition to Lyapunov stability is as follows—Consider the map:

$$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$$

defined by $\Phi_t(x_0) = x(t, x_0)$. Then a system is said to be stable in the sense of Lyapunov at the equilibrium point x_e if and only if Φ_t is continuous at x_e, for each $t \geq t_0$.

February 19, 2019

We've covered a high-level overview of Lyapunov Stability and covered some of the math required for understanding it. Now we define Stability in the Sense of Lyapunov and Asymptotic Stability in the context of Lyapunov Stability, and cover Lyapunov's Direct Method for proving stability.

Definition 2.5. *Let x_e be an equilibrium point of $\dot{x} = f(x)$ such that there exists a $\rho > 0$ for which $x_0 \in B_\rho(x_e)$. This implies that $x(t, x_0)$ exists for all $t \geq 0$ and is unique.*

a. x_e is SISL if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $\|x_0 - x_e\| < \delta$ implies that $\|x(t, x_0) - x_e\| < \epsilon,\ \forall t \geq \delta$

b. x_e is unstable otherwise

c. x_e is asympotically stable if it is SISL and $\exists \nu > 0$ such theat $\|x_0 - x_e\| < \nu$ implies that $\lim_{t\to\infty} \|x(t, x_0) - x_e\| = 0$

Remark. Asymptotic Stability is SISL plus Attractivity

Theorem 2.6 (Lyapunov's Direct Method)**.** *Assume $x_e = 0$ is an equilibrium point of $\dot{x} = f(x)$ (a time invariant function), and that there exists an open set $\mathcal{D}$ about the origin such that*

1. $f : \mathcal{D} \to \mathbb{R}^n$ is locally Lipschitz

2. There exists a continuously differentiable (C^1) function $V : D \to \mathbb{R}$ such that

(a) $V(0) = 0$

(b) $V(x) > 0$, $\forall x \in \mathcal{D}$, $x \neq 0$

(c) $\dot{V}(x) \leq 0$, $\forall x \in \mathcal{D}$

Then $x_e = 0$ is SISL. Additionally, if

(d) $\dot{V}(x) < 0$, $\forall x \neq 0, x \in \mathcal{D}$

Then $x_e = 0$ is an asymptotically stable equilibrium point.

2.1 Proof of Lyapunov's Direct Method

This proof is one of the major foundations of nonlinear control theory. We divide the proof into two parts. First, we prove that parts 1 and 2abc from 2.6 imply that the system is SISL. Then we prove that a SISL system satisfying 2d from 2.6 is Asymptotically Stable.

Part 1: SISL

We want to show that

1. $\exists \rho > 0$ such that $\forall x_0 \in B_\rho(0)$ implies that $x(t, x_0)$ exists on $[0, \infty)$ and is unique
2. $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|x\| < \delta$ implies that $\|x(t, x_0)\| < \epsilon$, $\forall t \geq 0$

Our proof combines both. We'll first define and prove three claims, then combine them to prove SISL. Let $\epsilon > 0$. Then we can choose $0 < r \leq \epsilon$ such that

$$\bar{B}_r(0) := \{x \in \mathbb{R}^n | \|x\| \leq r\} \subset \mathcal{D} \tag{2.1}$$

(Since $\mathcal{D}$ is open) Explain further

PIC HERE

Let's define α

$$\alpha := \min_{\|x\|=r} V(x) > 0 \tag{2.2}$$

We know that α exists because V is continuous and $\|x\| = r$ is a compact set. We also know that $\alpha > 0$ because $V(x) > 0, x \neq 0$. Then we can define a β where $0 < \beta < \alpha$ and define

$$\Omega_\beta := \{x \in \bar{B}_r(0) | V(x) \leq \beta\} \tag{2.3}$$

By construction $\Omega_\beta \subset \bar{B}_r(0)$.

Claim 0: $\Omega_\beta \subset B_r(0)$

Proof. Suppose $x \in \Omega_\beta$ and $\|x\| = r$. Then $V(x) \geq \alpha$ (because $\alpha = \min_{\|x\|=r} V(x)$). But $x \in \Omega_\beta \implies V(x) \leq \beta < \alpha$. This is a contradiction. Thus, claim 0 is proved. ■

Claim 1: Solutions that start in Ω_β stay in Ω_β. If this is true, then by 2.3 $\forall x_0 \in \Omega_\beta$, solutions exist on $[0, \infty)$ and are unique.) MISSED EXPLANATION

Proof. Let $\phi(t)$ be a solution of $\dot{x} = f(x)$ defined on $[t_1, t_2]$ with $\phi(t_1) \in \Omega_\beta$. We need to show that

$$\phi(t) \in \Omega_\beta \text{ for } t_1 \leq t \leq t_2 \iff V(\phi(t)) \leq \beta \text{ for } t_1 \leq t \leq t_2. \tag{2.4}$$

We have $\dot{V} \leq 0, \forall x \in \mathcal{D}$. Then $V(\phi(t)) \leq V(\phi(t_1)) \leq \beta$, $\forall t \in [t_1, t_2]$. Ie, V is non-increasing. Thus, since $V(x)$ starts in Ω_β it will stay there. ■

Claim 2: $\exists \delta > 0$ such that $B_\delta(0) \subset \Omega_\beta$

Proof. $V(x)$ is continuous. Thus

$$\begin{gathered}
x_e = 0 \implies \exists \delta > 0 \text{ such that } \|x - 0\| < \delta \implies \|V(x) - V(0)\| < \beta \\
\iff \\
\|x\| < \delta \implies \|V(x)\| < \beta \text{ (since } V(0) = 0) \\
\iff \\
\|x\| < \delta \implies V(x) < \beta \text{ (since } V(x) \geq 0 \forall x) \\
\implies \\
B_\delta(0) \subset \Omega_\beta
\end{gathered}$$

■

We put the claims together:

$$\begin{gathered}
\|x_0\| < \delta \implies x_0 \in \Omega_\beta \text{ (Claim 2)} \\
x_0 \in \Omega_\beta \implies x(t, x_0) \text{ exists on } [0, \infty) \text{ is unique and remains in } \Omega_\beta \text{ (Claim 1)} \\
\text{This implies that } x(t, x_0) \in B_\epsilon(0) \text{ (Claim 0)}
\end{gathered}$$

Remember that $\dot{V} = L_f V = \frac{\partial V}{\partial x} f(x)$

Part 2:

Now assume $\dot{V}(x) < 0 \forall x \neq 0, x \in \mathcal{D}$. We want to show that $\exists \nu > 0$ such that $\|x_0\| < \nu \implies \lim_{t\to\infty} \|x(t, x_0)\| = 0$. We'll show that $\nu = \delta$ works (where δ is from Claim 2).

Claim 3: If $\|x_0\| < \delta$, then $\lim_{t\to\infty} V(x(t, x_0)) = 0$

Proof.

$$\begin{gathered}
\|x_0\| < \delta \implies x(t, x_0) \in \Omega_\beta \forall t \geq 0 \\
\dot{V}(x(t, x_0)) \leq 0, \ \forall t \geq 0 \implies V(x(t, x_0)) \text{ is non-increasing} \\
V(x(t, x_0)) \geq 0 \implies \exists c \in \dot{\mathbb{R}} \text{ such that } \lim_{t\to\infty} V(x(t)) = c
\end{gathered}$$

■

We now use this to show that $c = 0$. Suppose $c \neq 0$. Then $V(x(t, x_0)) \geq \frac{c}{2}$, $\forall t \geq 0$. Then $x(t, x_0)$ never enters $\Omega_{\frac{c}{2}}$. Since $V(x)$ is continuous at $x = 0$, $\exists d > 0$ such that $B_d(0) \subset \Omega_{\frac{c}{2}}$. Thus, $x(t, x_0)$ is contained in the region shown here:

$$-\gamma := \max_{d \leq \|x\| \leq r} \dot{V}(x) < 0 \tag{2.5}$$

Proof.

$$V(x(t,x_0)) = V(x_0) + \int_{t_0}^{t} \dot{V}(x(\tau,x_0))d\tau \tag{2.6}$$

$$\leq V(x_0) + \int_{t_0}^{t} -\gamma d\tau \tag{2.7}$$

$$= V(x_0) - \gamma(t-t_0) \to -\infty \text{ as } t \to \infty \tag{2.8}$$

This is a contradiction. Thus

$$c = 0 \implies V(x(t,x_0)) \to 0 \text{ as } t \to \infty \iff x(t,x_0) \to 0 \text{ as } t \to \infty \tag{2.9}$$

■

February 21, 2019

Theorem 2.7 (Lyapunov's Direct Method)**.** *Given $\dot{x} = f(x)$, $f(0) = 0$ and there exists an open nbd????? $\mathcal{D}$ about $x = 0$ such that*

1. *$f : \mathcal{D} \to \mathbb{R}^n$ is locally Lipschitz*
2. *$\exists V : \mathcal{D} \to \mathbb{R}$, $V \in C^1$ such that*
   (a) *$V(0) = 0$*
   (b) *$V(x) > 0,\ \forall x \in D,\ x \neq 0$*
   (c) *$\dot{V}(x) \leq 0,\ \forall x \in \mathcal{D}$*
   (d) *$\dot{V}(x) < 0,\ \forall x \in \mathcal{D}\ x \neq 0$*

If cases abc are fulfilled, it's SISL. If abcd, it's AS.

Theorem 2.8. *(Converse Lyapunov Theorem) For all the Lyapunov theorems, there exist converse theorems. If a function is SISL, there must exist a Lyapunov function which shows that it's SISL. If a function is AS, there must exist a Lyapunov function which shows that it's AS. And so on.*

Definition 2.9 (Globally Asymptotic Stability (GAS))**.** *$x_e = 0$ is GAS if it is SISL and $\forall x_0 \in \mathbb{R}^n$, $x(t,x_0)$ exists on $[0,\infty)$ and MISSED THE REST*

Theorem 2.10. *Let $x_e = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous differentiable function such that*

1. *$V(0) = 0$, $V(x) > 0,\ \forall x \neq 0$*
2. *$V(x) \to \infty$ as $\|x\| \to \infty$ (It's must be radially unbounded)*
3. *$\dot{V}(x) < 0,\ \forall x \neq 0$*

Then $x_e = 0$ is GAS.

Why must V be radially unbounded? It captures the notion of observability.

Example:

Let's take the Lyapunov function

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2 \tag{2.10}$$

Condition (a) holds, as $V(0) = 0$, and $V(x) > 0,\ x \neq 0$. Condition (c) holds, as we assume that $\dot{V}(x) < 0, \forall x \neq 0$
INSERT PICTURE HERE
As we can see, if you're farther away you won't converge to zero.

2.1.1 Example: Mass-Spring Damper

: Let's take a nonlinear mass spring damper system

$$m\ddot{x} + b\dot{x} + k(x) = 0 \tag{2.11}$$

INSERT PICTURE HERE
The spring has a nonlinear spring force k that looks like this
INSERT PICTURE HERE
We can see that $xk(x) > 0,\ x \neq 0$. Thus the function is Lyapunov for the disc $\|x\| < d$.

We can define our state as $q = [x, \dot{x}]^T$, and get

$$\dot{q}_1 = q_2 \tag{2.12}$$

$$\dot{q}_2 = -\frac{b}{m} q_2 - \frac{k(q_1)}{m} \tag{2.13}$$

In order to prove stability, we need to find a Lyapunov function satisfying the conditions of THEOREM. We suggest a candidate Lyapunov function equal to the total energy $KE + PE$. We know that

$$KE = \frac{1}{2} m q_2^2 \tag{2.14}$$

$$PE = \int_0^{q_1} k(\sigma) d\sigma \tag{2.15}$$

Thus

$$V(q) = \frac{1}{2} m q_2^2 + \int_0^{q_1} k(\sigma) d\sigma \tag{2.16}$$

Let's check our conditions:

Condition (a):

$$V(0, 0) = \frac{1}{2} m 0^2 + \int_0^0 k(\sigma) d\sigma = 0 \tag{2.17}$$

Since k is locally Lyapunov, we know that

$$V(q_1, q_2) > 0,\ (q_1, q_2) \neq 0,\ |q_1| < d,\ q_2 \in \mathbb{R} \tag{2.18}$$

Condition (b): We're only proving local stability, so we don't need this

Condition (c): We can find the derivative

$$\dot{V}(q_1, q_2) = \frac{\partial V}{\partial q_1}\dot{q}_1 + \frac{\partial V}{\partial q_2}\dot{q}_2 \tag{2.19}$$

$$= k(q_1)q_2 - bq_2^2 - k(q_1)q_2 \tag{2.20}$$

$$= -bq_2^2 \tag{2.21}$$

Since $\dot{V}$ only depends on q_2, it can be zero when q_1 is nonzero. Thus we only have SISL.

Now, we know that the system is a mass spring damper, so we know that it's asymptotically stable. But we only proved SISL. What went wrong? We need to either

- Try a different (better) Lyapunov function
- Prove a better theorem

Let's find a better Lyapunov function first. Let's try the function

$$W = \frac{1}{m}\int_0^{q_1} k(\sigma)d\sigma + [q_1, q_2]\begin{bmatrix} \frac{1}{2}\left(\frac{b}{m}\right)^2 & \frac{1}{2}\frac{b}{m} \\ \frac{1}{2}\frac{b}{m} & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{2.22}$$

We can see that

- $W(0,0) = 0$
- $W(q_1, q_2) > 0, \ \forall (q_1, q_2) \neq (0,0), \ |q_1| < d$
- $\dot{W}(q_1, q_2) = \frac{-b}{2m}q_1 k(q_1) - \frac{b}{2m}q_2^2 < 0, \ \forall (q_1, q_2) \neq (0,0)$

Thus we see it's asymptotically stable.

Now let's try to prove a better theorem:

Theorem 2.11 (LaSalle's Invariance Theorem (Corr 4.1))**.** *Let $x_e = 0$ be an equilibrium point of $\dot{x} = f(x)$, where f is locally Lipschitz. Assume that there exists an open set $\mathcal{D}$ containing $x_e = 0$ and a Lyapunov function $V : \mathcal{D} \to \mathbb{R}, \ V \in C^1$, such that*

1. $V(0) = 0, \ V(x) > 0, \ \forall x \in \mathcal{D}, \ x \neq 0$
2. $\dot{V}(x) \leq 0, \ \forall x \in \mathcal{D}$

Let $S = \{x \in \mathcal{D} | \dot{V}(x) = 0\}$, and suppose that the only function $\phi : [0,T] \to \mathbb{R}^n$ that satisfies

3. $\dot{\phi}(t) = f(\phi(t))$ *(which means that $\phi(t)$ is a solution to the differential equation)*
4. $\phi(t) \in S, \ 0 \leq t \leq T$ *is* $\phi(t) \equiv 0$.

Then $x_e = 0$ is asymptotically stable

Let's use this to check our first Lyapunov function, where $\dot{V}(q_1, q_2) = -bq_2^2$. We define

$$S = \{x \in \mathcal{D} | \dot{V}(q_1, q_2) = 0\} = \{x \in \mathcal{D} ||q_1| < d,\ q_2 \equiv 0\} \tag{2.23}$$

Where

$$\mathcal{D} = \{q \in \mathbb{R}^2 ||q_1| < d,\ q_2 \in \mathbb{R}\} \tag{2.24}$$

Let $\phi(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}$ be a solution that lies in S. Thus $\phi_2(t) \equiv 0$ for $|\phi_1(t)| < d,\ \forall t$.

MISSED THE REST

Definition 2.12 (Instability:)**.** *Let's take the system*

$$\dot{x} = f(x),\ f(0) = 0 \tag{2.25}$$

We say that $x_e = 0$ is unstable if

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0,\ \|x_0\| < \delta \implies \exists T > 0, \|x(T, x_0)\| \geq \epsilon \tag{2.26}$$

In other words, an equilibrium point is unstable if there exists some ball of radius ϵ such that for all balls, an initial condition within the ball will lead to a the function leaving the ball of radius ϵ.

2.1.2 Examples:

INCLUDE PICTURES

February 26, 2019

Last time we covered Global Asymptotic Stability, LaSalle's theroem, and a bit of instability. Today we conclude our coverage of instability and Quadratic Lyapunov functions.

Quick review of LaSalle's theorem:

Theorem 2.13. *W4e have $\dot{x} = f(x)$, $f(0) - 0$, and $\mathcal{D} \in \mathbb{R}$, with $0 \in \mathbb{D}$. We have a function $V : \mathcal{D} \to \mathbb{R}$, with $V \in C^1$ such that*

1. *$V(0) = 0$, $V(x) > 0$, $\forall x \in \mathcal{D}$, $x \neq 0$*
2. *$\dot{V} \leq 0$ on $\mathcal{D}$*

Let $S = \{x \in \mathcal{D} | \dot{V} = 0\}$ and suppose that the only solution in S is $x(t) \equiv 0$. Then $x_e = 0$ is asymptotically stable.

LaSalle's Invariance Principle

Theorem 2.14. *Let's take $S = \{x \in \mathcal{D} | \dot{V} = 0\}$ and let M be the largest invariant set in S, then M is asymptotically stable. If $M = \{0\}$, then 0 is asymptotically stable. If M is a larger set, you cannot say anything about the asymptotic stability of any of those points.*

Example: We have a harmonic oscillator

$$\ddot{x} + \dot{x} + \sin(x) = 0 \tag{2.27}$$

and define $x_1 = x$ and $x_2 = \dot{x}$. We then have

$$\dot{x}_1 = x_2 \tag{2.28}$$

$$\dot{x}_2 = -x_2 - \sin(x_1) \tag{2.29}$$

If we define $V = KE + PE$, we have

$$\dot{V} = -x_2^2 \leq 0 \tag{2.30}$$

We find an S

$$S = \{x \in \mathbb{R}^2 | \dot{V}(x_1, x_2) = 0\} = \{x_1 \in \mathbb{R},\, x_2 = 0\} \tag{2.31}$$

On S we have

$$\dot{x}_1 = 0 \tag{2.32}$$

$$\dot{x}_2 = -\sin x_1 = 0 \text{ when } x_1 = k\pi,\ K \in \mathbb{Z} \tag{2.33}$$

Now we define M

$$M = \{(k\pi), 0),\ k \in \mathbb{Z}\} \tag{2.34}$$

If we define $\mathcal{D}$ as

$$\mathcal{D} = \{(-\pi, \pi) \times \mathbb{R}\} \tag{2.35}$$

then

$$M = \{(0, 0)\} \tag{2.36}$$

Thus $(0, 0)$ is locally asymptotically stable.

2.1.3 Instability

Definition 2.15. *$x_e = 0$ is unstable if $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists x_0$ with $\|x_0\| < \delta$ and $T > 0$ such that $\|x(T, x_0)\| > \epsilon$*

We want to find a sufficient condition for an equilibrium point to be stable. Let's try

1. $V(0) = 0$
2. $V(x) > 0,\ \forall x \in \mathcal{D},\ x \neq 0$
3. $\dot{V} > 0,\ \forall x \in \mathcal{D},\ x \neq 0$

This is much stronger than required, since it states that the equilibrium point is anti-stable.

Example

:

$$\dot{x}_1 = -x_1 \tag{2.37}$$
$$\dot{x}_2 = x_2 \tag{2.38}$$

We find a Lyapunov function

$$W(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{2.39}$$

where

$$\dot{W}(x_1, x_2) = -x_1^2 + x_2^2 \tag{2.40}$$

This doesn't fulfill the above conditions, but we know that the system is unstable. We can look at this other function

$$\bar{V} = -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{2.41}$$

This isn't positive definite, but its derivative is. What does this mean? I have no idea

Theorem 2.16 (Chetaev's instability theorem)**.** *Let $x_e = 0$ be an equilibrium point of $\dot{x} = f(x)$. Suppose that there exists an open set $\mathcal{D}$ of the origin on which f is locally Lipschitz and that there exists a V on that open set $V : \mathcal{D} \to \mathbb{R}$, where $V \in C^1$ such that*

1. $V(0) = 0$
2. $\forall\delta > 0$, $\exists x_0 \in B_\delta(0)$ *such that* $V(x_0) > 0$
3. $\exists\epsilon > 0$ *such that* $\dot{V}(x) > 0$ *on*

$$U := \{x \in B_\epsilon(0) | V(x) > 0\} \subset \mathcal{D} \tag{2.42}$$

Then $x_e = 0$ is unstable.

2.1.4 Linearization and Quadratic Lyapunov functions

Let's look at the linear system $\dot{x} = Ax,\ x \in \mathbb{R}^n$. We'll define some terms

Definition 2.17 (Quadratic function)**.** *Let $P \in \mathbb{R}^{n\times n}$. A function*

$$V(x) = x^T P x \tag{2.43}$$

is a quadratic function

Definition 2.18 (Symmetric matrix)**.** *A matrix M is symmetric if $M^T = M$*

Definition 2.19 (Anti-Symmetric matrix)**.** *A matrix M is anti-symmetric (also called skew-symmetric) if $M^T = -M$*

We can define any matrix P as a sum of a symmetric and skew-symmetric matrix. We have

$$P = \left(\frac{P+P^T}{2}\right) + \left(\frac{P-P^T}{2}\right) \tag{2.44}$$

The first term is symmetric, while the second is skew-symmetric. Now let's use P to define a quadratic function. We maintain that

$$f(x) = x^T P x = x^T \left(\frac{P+P^T}{2}\right) x \tag{2.45}$$

Why is this? Since $x^T A x$ is a scalar,

$$x^T A x = (x^T A x)^T = x^T A^T x \tag{2.46}$$

Thus we need $A = A^T$ The only way that this happens for a skew-symmetric matrix is if $A = A^T = 0$. Thus, we can assume that for every quadratic function $P = P^T$.

Definition 2.20 (Positive Definite Function)**.** *$V(x) = x^T P x$ is a positive definite function if $x^T P x > 0, \forall x \neq 0$. One says that P is a positive definite matrix and write this as $P \succ 0$. If P is symmetric (and we usually assume that it is), then if $P \succ 0$, $\lambda_i\{P\} > 0$. Another test: Missed This*

Definition 2.21 (Negative definite matrix)**.** *A symmetric matrix Q is negative definite if $-Q \succ 0$, which implies that $\lambda_i\{Q\} < 0$*

2.1.5 Lyapunov Equation

If we define $\dot{x} = Ax$ and $V(x) = x^T P x,\ P \succ 0$. Then

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} && (2.47)\\ &= x^T A^T P x + x^T P A x && (2.48)\\ &= x^T (A^T P + PA) x = -x^T Q x,\ Q \prec 0 && (2.49) \end{aligned}$$

When $\exists P \succ 0$ and Q

Missed This

February 26th, 2019, Frank's Version

Theorem 2.22 (**LaSalle's Invariant Principle**)**.** *Let $S \equiv \{x \in D | \dot{V}(x) = 0\}$, and let M be the largest invariant set in S. Then M is asymptotically stable.*

Example. Consider the simple harmonic oscillator given by:

$$\ddot{x} + \dot{x} + \sin x = 0,$$

$$\Rightarrow \begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_2 - \sin x_1 \end{cases}$$

and consider the candidate Lyapunov function $V(x_1, x_2)$ defined using the total energy (kinetic + potential) in the system:

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + 1 - \cos x_1,$$
$$\Rightarrow \dot{V}(x_1, x_2) = x_2(x_2 - \sin x_1) + \sin x_1 \cdot x_2 = -x_2^2 \leq 0.$$

By definition, the set S is contained in the zero level set of V, as follows:

$$S \equiv \{x \in \mathbb{R}^2 | V(x) = 0\} = \mathbb{R} \times \{0\},$$

and the dynamics on S become:

$$\dot{x}_1 = 0,$$
$$\dot{x}_2 = -\sin x.$$

Observe that x_1 and $\dot{x}_2$ are constant. Thus, we have:

$$M = \{(k\pi, 0) | k \in \mathbb{Z}\}$$

Thus, if we constrain $D = (-\pi, \pi) \times \{0\}$, then $M = \{(0, 0)\}$ is asymptotically stable.

Definition 2.23 (**Unstable Equilibrium Point**). *Let $\Sigma : \dot{x} = f(x)$ be a system with an equilibrium at 0. The equilibrium point 0 is said to be* ***unstable*** *if there exists some $\epsilon > 0$, such that for each $\delta > 0$, there exists some $x_0 \in B_\delta(0)$ and $t > 0$ such that:*

$$\|x(t, x_0)\| \geq \epsilon$$

.

Definition 2.24 (**Anti-Stable Equilibrium Point**). *An equilibrium point is called* ***anti-stable*** *if there exists some $\epsilon > 0$ such that, for each $x_0 \in \mathbb{R}^n$:*

$$\|x(t, x_0)\| \geq \epsilon$$

for each t.

Below, we wish to construct a set of sufficient conditions that guarantee an equilibrium point to be unstable. Consider the example below.

Example. Consider the (linear) system given by:

$$\dot{x}_1 = -x_1,$$
$$\dot{x}_2 = x_2.$$

The total energy function is thus:

$$W(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2,$$
$$\Rightarrow \dot{W}(x_1, x_2) = x_1\dot{x}_1 + x_2\dot{x}_2 = -x_1^2 + x_2^2.$$

Since $\dot{W}$ is not neither non-negative nor non-positive for all x, Lyapunov's Stability Theorems do not provide any information regarding the stability of the equilibrium point $(x_1, x_2) = (0, 0)$.

However, from linear system theory, we know that this system is unstable. However, consider the function $\overline{V}(x_1, x_2)$, given by:

$$\overline{V}(x_1, x_2) = -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2,$$
$$\dot{\overline{V}}(x_1, x_2) = x_1^2 + x_2^2 \geq 0,$$

Observe that $\dot{\overline{V}}(x_1, x_2) > 0$ for each $x \neq 0$. Geometrically, the level sets of $\overline{V}$ (hyperbolas) show x to be unstable if $x_{20} \neq 0$.

Theorem 2.25 (**Chetaev's Instability Theorem**)**.** *Let $x_e = 0$ be an equilibrium point of the system $\dot{x} = f(x)$. Suppose there exists some open set D, containing 0, such that f is locally Lipschitz on D, and some C^1 function $V : D \to \mathbb{R}$, such that:*

- *$V(0) = 0$.*
- *For each $\delta > 0$, there exists some $x_0 \in B_\delta(0)$ such that $V(x_0) > 0$.*
- *There exists some $\epsilon > 0$ such that $\dot{V}(x) > 0$ on the set:*

$$U \equiv B_\epsilon(0) \cap V^{-1}\{(0, \infty)\}$$

Moreover, $U \subset D$.

Then $x_e = 0$ is unstable.

Definition 2.26 (**Symmetric, Anti-Symmetric, Positive Definite, Negative Definite Matrices**)**.** *A square matrix $P \in \mathbb{R}^{n \times n}$ is called:*

1. ***Symmetric**, if $P^T = P$,*
2. ***Anti-symmetric**, if $P^T = P$,*
3. ***Positive definite**, if $P^T = P$ and $\sigma(P) \subset \mathbb{R}^+$.*
4. ***Positive semi-definite**, if $P^T = P$ and $\sigma(P) \subset \overline{\mathbb{R}^+}$.*
5. ***Negative definite**, if $P^T = P$ and $\sigma(P) \subset \mathbb{R}^-$.*
6. ***Negative semi-definite**, if $P^T = P$ and $\sigma(P) \subset \overline{\mathbb{R}^-}$.*

*Here, $\sigma(P)$, the **spectrum** of P, denotes the set of all eigenvalues of P.*

The above definitions are associated with a number of commonly known and/or used results, e.g. a symmetric matrix P is negative definite if and only if $-P$ is positive definite. For more details, the reader is referred to [5]. Positive definiteness, positive semi-definiteness, negative definiteness, negative semi-definiteness are concepts closely related to that of the quadratic form, as defined below.

Definition 2.27 (**Quadratic Form**). *Let $P \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$. Then $V(x) = x^T Px$ is called a **quadratic form**.*

The following proposition explains why it suffices to study the quadratic form for symmetric P.

Proposition 2.28. *For any $P \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$:*

$$x^T Px = x^T \left(\frac{P + P^T}{2}\right) x$$

Proof. Since $x^T Px \in \mathbb{R}$, we have:

$$\begin{aligned} x^T Px &= (x^T Px)^T = x^T P^T x, \\ \Rightarrow x^T Px &= \frac{1}{2} \cdot 2x^T Px = \frac{1}{2}(x^T Px + x^T P^T x) = x^T \left(\frac{P + P^T}{2}\right) x \end{aligned}$$

■

Below, we define what it means for a *function* (positive definite) to be positive definite.

Definition 2.29 (**Positive Definite Function**). *The function $V(x) = x^T Px$ is a **positive definite function** if $V(x) > 0$ for each $x \neq 0$.*

Remark. Observe that, if $V(x) = x^T Px > 0$ for each $x \neq 0$, and $P^T = P$, then P is a positive definite matrix.

Now, consider the linear system $\dot{x} = Ax$, with the value function $V(x) = x^T Px$, where $P > 0$. Then:

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T Px + x^T P\dot{x} = (Ax)^T Px + x^T P(Ax) \\ &= x^T (A^T P + PA)x \end{aligned}$$

Thus, to show stability, we must demonstrate the existence of some $P, Q > 0$ such that:

$$Q \equiv -(A^T P + PA) > 0.$$

February 28th, 2019

Last time we looked at the Lyapunov conditions for instability, as well as quadratic Lyapunov equations. We primarily looked at this in the context of the Lyapunov Equation, which is used to prove stability of linear systems. For a system

$$\dot{x} = Ax,\ V(x) = x^T Px,\ P \succ 0 \tag{2.50}$$

We have

$$\dot{V}(x) = x^T (A^T P + PA)x =: -x^T Qx \tag{2.51}$$

We want to know when $\exists P \succ 0, Q \succ 0$ such that

$$A^T P + PA = -Q \tag{2.52}$$

Theorem 2.30. *Given an $n \times n$ real matrix A, the following are equivalent:*

1. *All eigenvalues of A have negative real parts (A is Hurwitz)*
2. *There exists some positive definite Q such that $A^T P + PA = -Q$ has a unique solution P where $P \succ 0$*
3. *For all $Q > 0$, the equation $A^T P + PA = -Q$ has a unique solution P where $P \succ 0$*

To solve the equation, we can then choose Q as an arbitrary positive definite matrix (such as I). Then we solve for P.

2.1.6 LaSalle's Theorem and Observability

Earlier, we stated that LaSalle's theorem captures the observability of the nonlinear system. We'll not illustrate this with a linear system. Note, you'll never need to actually use LaSalle's for a linear system, since if a system is asymptotically stable, you can just make Q positive definite and find out. However, it's useful here for illustrative purposes. Let's say that we set Q to be some positive semidefinite matrix. Thus

$$A^T P + PA = -Q \preceq 0 \tag{2.53}$$

This means that the system is SISL. Let's say that we set $Q = C^T C$, where

$$\dot{x} = Ax + Bu \tag{2.54}$$

$$y = Cx \tag{2.55}$$

In this case

$$\dot{V}(x) = -x^T Q x = -x^T C^T C x = -y^T y \tag{2.56}$$

Since Q is negative semidefinite, this means that for some x, $\dot{V}$ can equal 0. Since $y^T y = 0$, we know that $y = 0$. If the system is asymptotically stable, then we must have

$$\dot{V} = 0 \implies x = 0 \tag{2.57}$$

This means that we need

$$y = 0 \implies x = 0 \tag{2.58}$$

This is only the case if the system is observable.

2.1.7 Lyapunov's Indirect Method

We now show how these results can be extended to nonlinear systems. To do that, we'll need to linearize the nonlinear systems.

Linearization:

Let's say we have a system

$$\dot{x} = f(x), \; f(x_e = 0) = 0 \tag{2.59}$$

Then we have

$$A := \frac{\partial f}{\partial x}(x_e) \tag{2.60}$$

This is just the Jacobian of f. The linearization is thus

$$\dot{\bar{x}} = A\bar{x} \tag{2.61}$$

where

$$\bar{x} = x - x_e \tag{2.62}$$

A Taylor series expansion yields

$$f(x) = f(0) + Ax + R(x) \tag{2.63}$$

where $R(x)$ satisfies

$$\lim_{\|x\| \to 0} \frac{\|R(x)\|}{\|x\|} = 0 \tag{2.64}$$

Basically, we want $R(x)$ to satisfy that for all $\gamma > 0$, there exists a $\sigma > 0$ such that

$$x \in B_\sigma(0) \implies \|R(x)\| \leq \gamma \|x\| \tag{2.65}$$

Remember that the Cauchy-Schwartz Inequality states that

$$\forall x, y \in \mathbb{R}^n, \; x^T y \leq |x^T y| \leq \|x\|_2 \|y\|_2 \tag{2.66}$$

Theorem 2.31 (Lyapunov's Indirect Method)**.** *Consider $\dot{x} = f(x)$, where $f \in C^1$ and $f(x_e = 0) = 0$.*

1. *If the linearization of the system about x_e has only eigenvalues with negative real parts, then the nonlinear system is locally asymptotically stable about x_e.*
2. *If the linearization has at least one eigenvalue with a positive real part, then x_e is unstable for $\dot{x} = f(x)$*
3. *If the linearization has at least one eigenvalue with real part equal to zero, and all other eigenvalues have negative real parts, then no conclusion can be made.*

Let's start by proving the third condition by contradiction.

Proof. Let's examine two systems

$$f_1(x) = -x^3 \tag{2.67}$$

and

$$f_2(x) = x^3 \tag{2.68}$$

The linearizations of both of these systems are

$$\dot{x} = 0 \tag{2.69}$$

However, we know that the first system is asymptotically stable, while the second is unstable. Thus, we cannot make a conclusion about the stability if an eigenvalue is zero. ■

Let's now prove the first condition

Proof. Let's consider a system $f(x)$ with a linearization

$$\dot{x} = Ax \tag{2.70}$$

Let's state that A has only negative real eigenvalues, and is therefore Hurwitz. Now let's consider the quadratic Lyapunov function

$$V(x) = x^T Px \tag{2.71}$$

and differentiate along $\dot{x} = f(x) = Ax + R(x)$. Where A is the Jacobian of $f(x)$, and $R(x)$ are the remaining terms of the Taylor expansion. We take

$$\begin{aligned} L_f V &= \frac{\partial V}{\partial x} f && (2.72)\\ &= 2x^T PAx + 2x^T PR(x) && (2.73)\\ & && (2.74) \end{aligned}$$

We can rearrange $2x^T PAx$ to be $x^T[A^T P + PA]x$ (since $x^T Cx$ is a scalar, and therefore is equal to its transpose). Now this is just the Lyapunov equation. Since A is Hurwitz, we know that we can find some P such that $A^T P + PA = -I$. Ths we have

$$-x^T x + 2x^T PR(x) \tag{2.75}$$

Now we use Cauchy-Schwartz on $x^T PR(x)$.

$$x^T PR(x) \leq |x^T PR(x)| \leq \|x\|_2 \|PR(x)\|_2 \leq \|x\|_2 \|P\|_i \|R(x)\|_2 \tag{2.76}$$

When $x \in B_\sigma(0)$, we know that $\|R(x)\|_2 \leq \gamma \|x\|_2$. Thus

$$x^T PR(x) \leq \|x\|_2 \|P\|_i \|R(x)\|_2 \leq \gamma \|x\|_2 \|P\|_i \|x\|_2 = \gamma \|P\|_i \|x\|_2^2 = \gamma \|P\|_i x^T x \tag{2.77}$$

Going back to $\dot{V}(x)$, we know know that

$$\dot{V}(x) = -x^T x + 2x^T PR(x) \leq -(1 - 2\gamma \|P\|_i) x^T x \tag{2.78}$$

We choose γ such that

$$1 - 2\gamma \|P\|_i > 0 \tag{2.79}$$

or

$$\gamma < \frac{1}{2\|P\|_i} \tag{2.80}$$

Thus

$$\dot{V}(x) \prec 0, \ \forall x \neq 0, x \in B_\sigma(0) \tag{2.81}$$

Thus, the system is locally asymptotically stable with region of attraction σ. ■

The proof for the unstable case is similar. We have two cases

Case I. A has no eigenvalues with real part equal 0, and has at least one eigenvalue with real part greater than zero.

Proof. Our objective is to construct a $V(x)$ such that $\dot{V}(x) \succ 0$ and $V(x)$ takes on positive values arbitrarily near the origin.

First, we use a similarity transformation to break A into two parts, one with positive eigenvalues, and one with negative.

$$\bar{A} = MAM^{-1} = \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & \bar{A}_{22} \end{bmatrix} \tag{2.82}$$

$\bar{A}_{11}$ has negative eigenvalues, and $\bar{A}_{22}$ has positive eiganvalues. Thus $\bar{A}_{11}$ is Hurwitz and $-\bar{A}_{22}$ is Hurwitz. Thus we solve two equations

(a) Solve

$$\bar{A}_{11}\bar{P}_{11} + \bar{P}_{11}\bar{A}_{11}^T = I_1 \implies \bar{P}_{11} \prec 0 \tag{2.83}$$

(b) Solve

$$\bar{A}_{22}\bar{P}_{22} + \bar{P}_{22}\bar{A}_{22}^T = I_2 \implies \bar{P}_{22} \succ 0 \tag{2.84}$$

We'll continue next time. ■

Case II. A has at least one eigenvalue with real part equal to 0, and at least one with real part greater than 0.

Note: $V(x) = x^T Px$ is a valid Lyapunov function for all linearized systems. It's therefore also a valid Lyapunov function for the nonlinear system (at least locally). While better Lyapunov functions can be found, this should be your first attempt.

March 5th, 2019

Time-Varying Systems:

Below, we will discuss time-varying systems, i.e. systems whose dynamics change with time. One can think of a time-varying system as one in which the state is placed in a different time-invariant system for each infinitesimal time interval. We begin with the definition of an equilibrium point for a time-varying system.

Definition 2.32 (Equilibrium Point: Time-Varying Case). *Given a time-varying system* $\Sigma : \dot{x} = f(t, x), x(t_0) = x_0$, *the point* $x = x_e$ *is called an* ***equilibrium point*** *of* Σ *if* $f(t, x_e) = 0$ *for each* $t \geq t_0$.

Time-varying systems exhibit less predictable properties than time-invariant systems.

- Region of Attraction:

  The region of attraction of an asymptotically stable time-varying system (as defined shortly below) may depend on t_0. In the worst case scenario, the region of attraction may shrink to the trivial case $\{x_e\}$, as $t_0 \to \infty$. This is illustrated in Figure **INCOMPLETE** below.

- Rate of Convergence:

  Even if an asymptotically stable time-varying system is stable for each choice of t_0, the rate of convergence of the system may depend on t_0. This is illustrated in Figure **INCOMPLETE** below.

- Lyapunov's Stability Theorem:

  Even if the conditions in Lyapunov's Stability Theorem (for time-invariant systems) hold for a time-varying system, i.e. there exists some continuously differentiable function $V(t,x)$ satisfying $V(t,x(t)) > 0$, $\dot{V}(t,x) \leq 0$ for each $t \geq t_0$ and $x \neq 0$, the system may not be stable in the sense of Lyapunov. This indicates that the time-invariant version of Lyapunov's Stability Theorem, as stated above, must be modified to render it applicable to time-varying systems. A counterexample will be provided after definitions for different notions of stability (for time-varying systems) have been established.

Below, consider the system:

$$\dot{x} = f(t,x), x(t_0) = x_0 \tag{2.85}$$

Definition 2.33 (**Stable in the sense of Lyapunov**). *The equilibrium point $x = 0$ is called a* ***stable equilibrium point*** *of the system (2.85) if, for any $t_0 \geq 0$ and $\epsilon > 0$, there exists some $\delta(t_0, \epsilon)$ such that:*

$$|x_0| < \delta(t_0, \epsilon) \quad \Rightarrow \quad |x(t)| < \epsilon, \forall t \geq t_0,$$

where $x(t)$ is the solution to (2.85), starting from $x(t_0) = x_0$.

Definition 2.34 (**Uniformly Stable**).

1. *The state $x_e \equiv 0$ is called* ***uniformly stable*** *if, for each $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, the mapping:*

   $$x(t) = \Phi(t, t_0)x_0$$

   is bounded by some positive constant.

2. *The equilibrium point $x = 0$ is called a* ***uniformly stable equilibrium point*** *of the system if it achieves the criterion for stable equilibrium points, with some $\delta(\epsilon)$ that is independent of t_0.*

In essence, a stable (in the sense of Lyapunov) equilibrium point is uniformly stable if the associated upper bounds $\delta(t_0, \epsilon)$ for its norms never approach 0, i.e.:

$$\inf_{t_0 \in \mathbb{R}} \delta(t_0, \epsilon) > 0$$

Definition 2.35 (**Asymptotically Stable**). *The state $x_e \equiv 0$ is called* ***asymptotically stable*** *if:*

1. $x_e \equiv 0$ *is a stable equilibrium point of* (2.85), *and*
2. $x(t)$ *converges to 0, i.e.* $\lim_{t\to\infty} \Phi(t, t_0) = 0$. *If this condition is met,* $x = 0$ *is said to be* ***attractive***.

The reader may question whether it is necessary to specify the first condition "$x_e \equiv 0$ is a stable equilibrium point of (2.85)" if the second statement "$\lim_{t\to\infty} \Phi(t, t_0) = 0$" already holds true. The following example answers this question in the affirmative.

Example. Consider the dynamical system given by:

$$\dot{x}_1 = x_1^2 - x_2^2$$
$$\dot{x}_2 = 2x_1 x_2.$$

The phase portrait of this system indicates that, although all trajectories following this system tends to $x = 0$ as $t \to \infty$, those particularly close to the x-axis will initially move far away from the origin before returning. In fact, one can choose a sequence of trajectories, increasingly closer to being parallel to the x-axis, such that the maximum distance (in time) between each trajectory and the origin increases as the sequence progresses. In this sense, $x = 0$ is not stable, even though it is attractive.

Definition 2.36 (**Uniformly Asymptotically Stable**). *The state $x_e \equiv 0$ is called* ***uniformly asymptotically stable*** *if:*

1. $x_e \equiv 0$ *is a uniform stable equilibrium point of* (2.85), *and*
2. $x(t)$ *converges uniformly to 0, i.e.* $\exists \delta > 0$, *and* $\gamma(\tau, x_0) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ *such that, whenever* $|x_0| < \delta$*:*

$$\|\phi(t, t_0)\| \leq \gamma(t - t_0, x_0)$$
$$\lim_{\tau\to\infty} \gamma(\tau, x_0) = 0$$

Let $\phi(t, x_0, t_0)$ *denotes the trajectory of the system* $\dot{x} = f(x, t), x(t_0) = t_0$, *starting from* x_0 *at time* t_0. *Then the second condition above is equivalent to the following statement—* $\exists \delta$ *and some non-decreasing function* $T : \mathbb{R}^+ \to \mathbb{R}^+$ *such that, whenever* $|x_0| < \delta$*:*

$$|\phi(t_1 + t, x_0, t_1)| < \epsilon$$

for each $t_1 \geq t_0$.

The definitions of asymptotic stability do not quantify the speed of convergence of trajectories to the origin, e.g. $1/t$, $1/\sqrt{t}$, etc. However, there is a particularly strong form of stability that demands an exponential rate of convergence.

Definition 2.37 (**Exponentially Stable, Rate of Convergence**). *The state $x_e \equiv 0$ is called **exponentially stable** with **rate of convergence** α if $x_e \equiv 0$ is stable, and $\exists M, \alpha > 0$ such that:*

$$\|x(t)\| \leq Me^{-\alpha(t-t_0)} \cdot |x_0|$$

Clearly, if a system is exponentially stable, it is uniformly asymptotically stable. We will later show that, for linear systems (whether time-invariant or time-varying), the converse is also true.

First, consider the following example, which shows that the time-invariant version of Lyapunov's stability theorem is not applicable to time-varying systems.

Example. Consider the system $\Sigma : \dot{x} = x, x(t) = x_0 \neq 0$. Clearly, this system is exponentially unstable. However, if we take $V(t,x) \equiv e^{-3t}x^2$, then we find that:

$$\begin{aligned} &V(t,0) = 0, \\ &V(t,x) > 0, \qquad \forall t, \forall x \neq 0. \dot{V(t,x)} = e^{-3t}x^2 \leq 0, \end{aligned}$$

with $\dot{V(t,x)} = 0$ if and only if $x = 0$. This appears to contradict Lyapunov's stability theorem. The reality is, however, that the previously given version of Lyapunov's stability theorem does not hold for time-varying systems.

In this particular example, the reason for this contradiction arises from the fact that as $t \to \infty$, we have $x(t) = e^t x_0 \to \infty$, but $V(t,x) = e^{-t}x_0^2 \to 0$. This indicates that, *to generalize the previously given version of Lyapunov's Stability Theorem to the time-varying case, we must upper bound $V(t,x)$ by some function of x that approaches ∞ when $|x| \to \infty$.*

The Basic Stability Theorem of Lyapunov, presented below, illustrates that the different definitions of stability mentioned above can be directly characterized by an energy function $V(x,t)$ that describes the system. This energy function is often upper and/or lower bounded by a set of continuous functions with particular properties. We first present definitions of broad classes of functions that satisfy these properties.

Definition 2.38 (**Classes of Functions, Part 1**).

1. *A function $\alpha(\cdot) : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ belongs to **class K**, denoted by $\alpha(\cdot) \in K$, if it is continuous, strictly increasing, and $\alpha(0) = 0$.*

2. *A function $\alpha(\cdot) : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ belongs to **class KR**, denoted by $\alpha(\cdot) \in K$, if $\alpha \in K$ and $\alpha(p) \to \infty$ as $p \to \infty$.*

Below, we characterize functions that behave locally and globally "like an energy function," in the sense that they increase in the direction away from the origin (which can, in the context of these definitions, be intuitively thought of as an attractive equilibrium point).

Definition 2.39 (**Classes of Functions, Part 2**).

1. *A function $v(x,t) : \mathbb{R}^n \times \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ is called* ***locally positive definite (l.p.d.)*** *if it is continuous, and there exists some $h > 0$ and some function $\alpha(\cdot) \in K$ such that:*

$$v(0,t) = 0,$$
$$v(x,t) \geq \alpha(|x|), \qquad \forall x \in B_h, \qquad t \geq 0$$

2. *A function $v(x,t) : \mathbb{R}^n \times \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ is called* ***(globally) positive definite (p.d.)*** *if it is continuous, and there exists some function $\alpha(\cdot) \in KR$ such that:*

$$v(0,t) = 0,$$
$$v(x,t) \geq \alpha(|x|), \qquad \forall x \in \mathbb{R}^n, \qquad t \geq 0$$

3. *A function $v(x,t) \to \mathbb{R}^n \times \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ is called* ***decrescent*** *if it is continuous, and there exists some function $\beta(\cdot) \in K$ such that:*

$$v(x,t) \leq \beta(|x|), \qquad \forall x \in B_h, \qquad t \geq 0$$

Remark. If $v(x,t)$ does not explicitly depend on the time t, it must be decrescent. This is because it is either bounded above by a function of class K, or unbounded above, in which case it is bounded by itself. In addition, if $v(x,t)$ is decrescent, then $v(0,t) \leq \beta(0) = 0$. (The equality follows from $\beta(\cdot) \in K$).

Examples are given below for each of the above types of functions.

Example (**Examples of l.p.d., p.d., and decrescent functions**). Here are some examples of energy-like functions and their membership in the various classes introduced above. It is an interesting exercise to check the appropriate functions of class K and KR that can to be used to verify these properties.

For the examples below, P is positive definite, whereas Q is not. No other information is assumed about P or Q.

Table 2.1: Classification of Value Functions

	$v(x,t)$	l.p.d.f.	p.d.f.	Decrescent
(1)	$\lvert x^2 \rvert$	Yes	Yes	Yes
(2)	$x^T P x$	Yes	Yes	Yes
(3)	$(t+1)\lvert x\rvert^2$	Yes	Yes	No
(4)	$e^{-t}\lvert x\rvert^2$	No	No	Yes
(5)	$\sin^2(\lvert x\rvert^2)$	Yes	No	Yes
(6)	$e^t x^T Q x$	No	No	No
(7)	$x_1^2 + x_2^4$	Yes	Yes	Yes
(8)	x_1^8	No	No	Yes
(9)	$(x_1 + x_2)^4$	No	No	Yes
(10)	$x_1^2 + (\sin x_2)^2$	No	No	Yes
(11)	$\frac{1}{1+t}(x_1^2 + x_2^2)$	No	No	Yes

(Examples (1)-(6) can be found in Professor Sastry's text [10], while Examples (7)-(11) are from Professor Sreenath's notes.)

The theorem below illustrates how imposing an increasingly strict set of conditions on the value function $v(x,t)$ and its time derivative $\dot{v}(x,t)$, defined *along the trajectory of the system's state*, allows us to make increasingly stronger claims regarding the stability of the system. In particular, we define $\dot{v}(x,t)$ as:

$$\begin{aligned} \frac{dv}{dt}(x,t)\bigg|_{\substack{\dot{x}=f(x,t)\\ x(t_0)=x_0}} &= \frac{\partial v}{\partial t}(x,t) + \frac{\partial v}{\partial x}(x,t)\frac{dx}{dt} \\ &= \frac{\partial v}{\partial t}(x,t) + \frac{\partial v}{\partial x}(x,t)f(x,t) \end{aligned}$$

This is called the *Lie derivative* of $v(x,t)$ *along* $f(x,t)$.

Theorem 2.40 (**Basic Lyapunov Theorems**). *Sets of conditions on $v(x,t)$ and $\dot{v}(x,t)$ are associated with notions of internal stability as given in the following table. Without loss of generality, we have placed the equilibrium point at the origin.*

Table 4.1

Table 2.2: Basic Lyapunov Theorems

	Conditions on $v(x,t)$	Conditions on $-\dot{v}(x,t)$	Conclusions
1	l.p.d.f.	≥ 0 locally	stable
2	l.p.d.f., decrescent	≥ 0 locally	unif. stable
3	l.p.d.f., decrescent	l.p.d.f.	unif. asymp. stable
4	p.d.f., decrescent	p.d.f.	globally unif. asymp. stable

March 7th, 2019

Last time we covered time-varying systems. This time we cover Lyapunov results for time varying systems and exponential stability, and do a review.

Arrived late and missed the first part

2.1.8 Exponential Stability

Consider $\dot{x} = f(t,x),\ f(t,0) = 0 \forall t \geq t_0$

Definition 2.41. *$x = 0$ for $\dot{x} = f(t,x)$ is exponentially stable if $\forall t \geq t_0$, there exists a $\sigma > 0$, a $\gamma > 0$, and an $N < \infty$ such that*

(i) $\forall x_0 \in B_\sigma(0)$, the solution $x(t,t_0,x_0)$ exists and is unique

(ii) $\|x(t, t_0, x_0)\| \le N\|x_0\|e^{-\gamma(t-t_0)}$, $\forall t \ge t_0$, $\forall x_0 \in B_\sigma(0)$

Remark: If σ, γ, N can be chosen independently of t_0, then $x_{eq} = 0$ is uniformly exponentially stable.

Why do we want exponential stability instead of asymptotic stability? Rate of convergence and robustness. With exponential stability we can assign a rate of convergence. Also, small linear perturbations on an asymptotically stable system will distabilize it, while small linear perturbations on an exponentially stable system will not.

Theorem 2.42. *Given the system $x = f(t, x)$, $f(t, 0) = 0$, $\forall t \ge t_0$, where f is piecewise continuous in t and Lipschitz continuous in x. Then a sufficient condition for $x = 0$ to be uniformly exponentially stable is that there exists some V where $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and there exist constants $\alpha, \beta, \gamma, r > 0$ such that $\forall x \in B_r(0)$ and $\forall t \ge t_0$.*

(a) $\alpha x^T x \le V(t, x) \le \beta x^T x$

(b) $\dot{V}(t, x) \le -\gamma V(t, x)$

This uses Lyapunov's direct method. You can also use the indirect method to get

Theorem 2.43 (Key to all Linear Control Theory)**.** *Consider a time-invariant system $\dot{x} = f(x)$, $f(0) = 0$. Then $x = 0$ is exponentially stable if and only if all eigenvalues of the jacobian $A = \frac{\partial F}{\partial x}(0)$ have negative real parts.*

Note that this means that all nonlinear systems that are locally asymptotically stable are also locally exponentially stable, unless their linearizations have at least one zero eigenvalue.

2.2 Review

2.2.1 Level Sets

Given some Lyapunov function $V : \mathcal{D} \to \mathbb{R}$ where $V(0) = 0$, and $V(x) > 0$, $\forall x \ne 0$, and given some ball $B_r(0) \in \mathcal{D}$, inside which its derivative $\dot{V}(x) < 0$, $\forall x \ne 0$, is the function Lyapunov stable for all points $x_0 \in B_r(0)$?

No it's not. $\dot{V}(x) < 0$ implies that x will move into smaller and smaller level sets of V. However, there's nothing saying that $B_r(0)$ is a level set of V. If the level sets of V extend outside of $B_r(0)$, then x may evolve outside of $B_r(0)$. Since x is outside $B_r(0)$ $\dot{V}(x)$ is no longer guaranteed to be negative, so the function could be unstable.

2.2.2 Global Asymptotic Stability using LaSalle's theorem

Let's say we have $V \succ 0$, $V(0) = 0$, $V \in C^1$, $V : \mathcal{D} \to \mathbb{R}$, and $\dot{V}(x) \le 0$. We define an

$$S = \{x \in \mathcal{D} | \dot{V}(x) = 0\} \tag{2.86}$$

If $x(t) \equiv 0$ is the only solution in S, then $x = 0$ is asymptotically stable. If V is radially unbounded, $x = 0$ is globally asymptotically stable.

2.2.3 Types of continuity

We have the following hierarchy

1. C^0
2. differentiable
3. C^1
4. $\dot{f}(x)$ is bounded
5. Lipschitz continuous
6. Uniformly continuous

END OF MIDTERM RANGE

Chapter 3

Feedback Control:

March 14th, 2019

Control Lyapunov Functions for Asymptotic Stability:

Definition 3.1 (**Full-State Feedback Control**)**.** *Given $\dot{x} = f(x,u)$, $f(0,0) = 0$, with $x \in \mathbb{R}^n, u \in \mathbb{R}^m$. We seek a **full-state feedback control** $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ such that $x_e = 0$ is **globally asymptotically stable** for the closed-loop function $\dot{x} = f\big(x, \alpha(x)\big)$.*

Theorem 3.2 (**Converse Lyapunov Theorem**)**.** *If there exists a solution to the above problem, then there exists some continuously differentiable $V : \mathbb{R}^n \to \mathbb{R}$ such that:*

- $V(0) = 0, V(x) > 0 \forall x \neq 0,$
- *V is radially unbounded,*
- *$\dot{V}(x) < 0$ for each $x \neq 0$.*

In particular, the following two statements are equivalent:

$$\forall x \neq 0, \dot{V}(x) = \frac{\partial V}{\partial x} f\big(x, \alpha(x)\big) < 0,$$
$$\Leftrightarrow \forall, \exists u = \alpha(x) \in \mathbb{R}^m \text{ such that } \frac{\partial V}{\partial x} f(x,u) < 0.$$

Definition 3.3 (**Control Lyapunov Function, CLF**)**.** *A **Control Lyapunov Function** (CLF) is a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying:*

1. *V is radially unbounded.*
2. $\inf\limits_{u \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial x} \cdot f(x,u) \right\} < 0.$

Remark. This is the globally asymptotic stability version of the definition. There exist other versions, e.g. for asymptotic stability or exponential stability.

Theorem 3.4. *Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a CLF for the system $\Sigma : \dot{x} = f(x, u), x \in \mathbb{R}^n, u \in \mathbb{R}^m$. Then there exists an infinitely continuously differentiable (i.e. smooth) feedback control $\alpha : \mathbb{R}^n \to \mathbb{R}^m$, such that the closed-loop system $\dot{x} = f\big(x, \alpha(s)\big)$ is globally asymptotically stable.*

Remark. The proof of this theorem is not constructive; i.e. it gives the existence of such a smooth feedback controller, but it does not tell us what this controller is. We will see theorems below that rectify this.

Definition 3.5 (**Control Affine System**)**.** *A* ***control affine system*** *is a system of the form:*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + g(x)u$$

where $x \in \mathbb{R}^n, u_i \in \mathbb{R}, u \in \mathbb{R}^m, g_i : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, with:

$$g(x) = \begin{bmatrix} g_1(x) & \cdots & g_m(x) \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

Definition 3.6 (**Lie Derivative**)**.** *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and let $\dot{x} = f(x)$. Then the* ***Lie derivative of*** *h* ***with respect to*** *f is defined as:*

$$L_f h(x) \equiv \frac{\partial h}{\partial x} f(x).$$

Example. For a control affine system:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i,$$

the value function $V(x)$ evolves as:

$$\begin{aligned}\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}\Big[f(x) + \sum_{i=1}^{m} g_i(x)u_i\Big] &= L_f V(x) + \sum_{i=1}^{m} L_{g_i}(x)u_i \\ &= L_f V(x) + L_g V(x)u.\end{aligned}$$

Proposition 3.7. *The following statements are equivalent:*

- *For each $x \neq 0$:*
$$\inf_{u\in\mathbb{R}^m} L_f V(x) + L_g V(x)u < 0.$$
- *For each $x \neq 0$, if $L_g V(x) = 0$, then $L_f V(x) < 0$.*

Proof. If $L_gV(x) = 0$, then the first statement is true if and only if, for each $x \neq 0$:

$$\inf_{u \in \mathbb{R}^m} L_fV(x) < 0$$

which in turn is true if and only if the second statement is true.

If $L_gV(x) \neq 0$, then, based on the values of $L_fV(x)$ and $L_gV(x)$, one can always choose u such that $L_fV(x) + L_gV(x)u$ is as negative as possible, i.e.:

$$\inf_{u \in \mathbb{R}^m} L_fV(x) = -\infty < 0$$

■

Example. Consider a control affine system $\Sigma : \dot{x} = f(x) + g(x)u$. If $V(x)$ is a control affine function for Σ, then we should choose $u^\star$ as follows:

$$u^\star = \begin{cases} 0, & L_fV < 0, \\ -(L_gV)^{-1}L_fV, & \text{else} \end{cases},$$

i.e. $u^\star$ solves the constrained optimization problem:

$$\begin{aligned} &\text{Minimize } u^Tu \\ &\text{subject to: } L_fV + L_gV \cdot u \leq 0. \end{aligned}$$

March 19th, 2019

For multiple inputs $\{u_i | i = 1, \cdots, m\}$, the min-norm controller becomes:

$$u^* = \begin{cases} -\frac{L_fV(x)}{L_gV(x)L_gV(x)^T} L_gV(x)^T, & L_fV(x) < 0 \\ 0, & \text{else} \end{cases}$$

Controlling the norm of u is very important. Controls naively designed to set $\dot{x}$ to a non-positive quantities may not have bounded norm throughout the domain of x, as the following example indicates.

Example. Consider the system $\Sigma : \dot{x} = x + x^2u$, where $x, u \in \mathbb{R}$. If we want $x = 0$ to be asymptotically stable, then we need:

$$\text{sgn}(\dot{x}) = \text{sgn}(x + x^2u) = -\text{sgn}(x).$$

In this example, this occurs if and only if:

$$\begin{aligned} u &< -\frac{1}{x}, & x > 0, \\ u &> -\frac{1}{x}, & x < 0. \end{aligned}$$

In particular, marginal stability holds at $u = 1/|x|$. Observe that $|u^\star| \to \infty$ as $x \to 0$. We say that the system has the *large control property*. Intuitively, this means that the control effort at x required to drive the system to 0 increases as $|x|$ increases.

As the above example illustrates, it is undesirable for a system to have the large control property. We give a definition for the opposite property below.

Definition 3.8 (**Small Control Property**). *A control Lyapunov function satisfies the* ***small control property*** *if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that for each $x \in B_\delta(0)$, there exists some $u(x) \in \mathbb{R}^m$ satisfying:*

1. $\|u\| < \epsilon$.
2. $\dot{V}(x,u) = L_fV(x) + L_gV(x)u < 0$.

Theorem 3.9 (**Sontag 1989, single input case**). *Suppose V is a CLF for the single-input system $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ where $f(0) = 0$ and $f(\cdot), g(\cdot)$ are Lipschitz continuous. Then, if the control $u(x)$ is chosen to be:*

$$\alpha_s(x) = \begin{cases} -\frac{L_fV(x)+\sqrt{L_fV(x)^2+L_gV(x)^4}}{L_gV(x)}, & L_gV \neq 0, \\ 0, & L_gV = 0. \end{cases}$$

Then the following statements are true:

1. $\dot{V}(x) = -\sqrt{L_fV(x)^2 + L_gV(x)^4} < 0$, *for each x, with $V(x) = 0$ if and only if $x = 0$. Moreover, $x = 0$ is globally asymptotically stable.*
2. $\alpha_s(\cdot)$ *is continuous for each $x \neq 0$.*
3. $\alpha_s(\cdot)$ *is continuous at $x = 0$ if $V(x)$ satisfies the small control property.*
4. $\alpha_s(\cdot) \in C^k$ *for each $x \neq 0$ if $V(\cdot) \in C^{k+1}$ and $f(\cdot), g(\cdot) \in \mathbb{C}^k$.*

Proof. (see Sontag's 1989 paper) ■

Example. We wish to design controls for the system:

$$\Sigma : \qquad \dot{x} = \sin x - x^3$$

using the following choice of control Lyapunov function:

$$V(x) = \frac{1}{2}x^2$$

First, let us check that $V(x)$ satisfies all the conditions in the definition of a control Lyapunov function. From its definition, $V \in C^1$, and V is positive definite and radially unbounded. It remains to check whether, for each $x \neq 0$ such that $L_gV(x) = 0$, we have $L_fV(x) < 0$. Evaluating $\dot{V}$, we have:

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x}\dot{x} = x(\sin x - x^3 + xu) = x\sin x - x^4 + x \cdot u, \\ \Rightarrow L_fV(x) &= x\sin x - x^4, \\ L_gV(x) &= x. \end{aligned}$$

Since $L_gV(x) = x$ is zero if and only if $x = 0$, the final condition is automatically satisfied.

Consider the following controls applied to Σ with the CLF choice $V(x) = \frac{1}{2}x^2$.

1. $\alpha_s(x)$: (Sontag control)

   From the definition of Sontag control, we have:

$$\begin{aligned}\alpha_s(x) &= -\frac{x(\sin x - x^3) + \sqrt{x^2(\sin x - x^3)^2 + x^4}}{x}, \qquad \forall x \neq 0 \\ &= -(\sin x - x^3) - \text{sgn}(x) \cdot \sqrt{(\sin x - x^3)^2 + x^2}, \qquad \forall x \neq 0,\end{aligned}$$

   and $\alpha_3(0) = 0$. (In fact, in this particular case, $\alpha_3(x)$ is continuous at $x = 0$ and in C^∞ at each $x \neq 0$.

   In this case:

$$\dot{V}(x) = -\sqrt{x^2(\sin x - x^3)^2 + x^4} < 0, \qquad \forall x \neq 0,$$

   with equality if and only if $x = 0$.

2. $\alpha_1(x) = \sin x + x^3 - x$:

   We have, for the closed-loop system and $\dot{V}$:

$$\begin{aligned}\Sigma_{\text{CL},1}: &\qquad \dot{x} = -x, \\ \Rightarrow \dot{V}(x) &= -x^2 \leq 0,\end{aligned}$$

   with equality if and only if $x = 0$. Thus, the closed-loop system is globally asymptotically stable.

3. $\alpha_2(x) = \sin x - x$:

   We have, for the closed-loop system and $\dot{V}$:

$$\begin{aligned}\Sigma_{\text{CL},2}: &\qquad \dot{x} = -x^3 - x, \\ \Rightarrow \dot{V}(x) &= -x^4 - x^2 \leq 0,\end{aligned}$$

   with equality if and only if $x = 0$. Thus, the closed-loop system is globally asymptotically stable.

4. $\alpha_3(x) = -\sin x$:

   We have, for the closed-loop system and $\dot{V}$:

$$\begin{aligned}\Sigma_{\text{CL},3}: &\qquad \dot{x} = -x^3, \\ \Rightarrow \dot{V}(x) &= -x^4 \leq 0,\end{aligned}$$

   with equality if and only if $x = 0$. Thus, the closed-loop system is globally asymptotically stable.

5. $\alpha_4(x) = -x$:

   We have, for the closed-loop system and $\dot{V}$:

$$\Sigma_{\text{CL},4}: \qquad \dot{x} = \sin x - x^3 - x,$$
$$\Rightarrow \dot{V}(x) = x(\sin x - x) - x^4 \leq 0,$$

   with equality if and only if $x = 0$. Thus, the closed-loop system is globally asymptotically stable.

Remark. Observe that, while all five controllers render the resulting closed-loop system globally asymptotically stable, they do so in noticeably different ways. Whereas the norm of the Sontag control dies off when the magnitude of x becomes large, this is not true for $\alpha_1(x), \alpha_2(x)$, or $\alpha_4(x)$, all of which become unbounded as $x \to \pm\infty$. This is mainly because of the inability of these three controllers to harness the intrinsic stabilizing capabilities of the $-x^3$ term already present in the open-loop system (before the application of any control).

March 21th, 2019

Next, we will consider backstepping, a form of recursive feedback design pioneered by Peter Kokotovic in the 1990s. For each $i \in \mathbb{N}$, the i-th iteration of the algorithm does the following:

1. Treat x_{i+1} as a virtual control of x_i, and find a function of x_i, e.g. $f(x_i)$ such that $x_{i+1} = f(x_i)$ stabilizes the system.
2. Define the error state $z_i \equiv x_{i+1} - f(x_i)$. Then, as $t \to \infty$, we have $x_{i+1} \to f(x_i)$ if and only if $z_i \to 0$.
3. Rewrite the dynamics of x_i, x_{i+1} in terms of x_i, z_i.
4. Define an augmented CLF $V_{a,i}(x_i, z) = V_i(x_i) + V_{z,i}(z)$, where $V_i(x_i)$ is an original CLF associated with the system, and $V_{z,i}(z)$ is some suitable function of z. Evaluate $\dot{V}_a$.
5. Try to choose an input u such that $\dot{V}_a(x) \leq 0$ for each x, with equality if and only if $x = 0$. If such an input u can be selected, then (x_i, z) is globally asymptotically stable. If z has been chosen carefully, this may imply that (x_i, x_{i+1}) is globally asymptotically stable as well.
6. Rewrite V_a, u in terms of the original coordinate system.

Backstepping is best illustrated through examples. Consider the two systems below.

Example. Consider the linear system:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = u.$$

Use backstepping to find a suitable control u to stabilize the system.

Remark. One choice of stabilizing control is:

$$u = -k_1 x_1 - k_2 x_2,$$

with corresponding closed-loop system:

$$\Sigma : \dot{x} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} x.$$

Solution :

1. We can also consider x_2 as a virtual control input. If we can set:

$$x_2 = -c_1 x_2, \qquad c_1 > 0,$$

this drives $x_1 \to 0$ as $t \to \infty$. (For more complex dynamics, a Lyapunov function—e.g. $V(x) = \frac{1}{2}x^2$ can be used to test for stablity).

2. Next, we define an error state between the state x_2 and its desired value $-c_1 x_1$:

$$z \equiv x_2 - (-c_1 x_2)$$

Then, as $t \to \infty$, we have $x_2 \to -c_1 x_1$ if and only if $z \to 0$.

3. Our next task is to rewrite the dynamics in terms of x_1 and z. This is done below:

$$\begin{aligned} \dot{x}_1 &= x_2 = z - c_1 x_1, \\ \dot{z} &= \dot{x}_2 + c_1 \dot{x}_1 = c_1 z - c_1^2 x_1 + u. \end{aligned}$$

4. Next, consider the following augmented CLF $V_a(x, z)$, and its Lie derivative:

$$\begin{aligned} V_a &\equiv \frac{1}{2}x^2 + \frac{1}{2}z^2, \\ \Rightarrow \dot{V}_a &\equiv x\dot{x} + z\dot{z} = x_1(z - c_1 x_1) + z(c_1 z - c_1^2 x_1 + u) \\ &= -c_1 x_1^2 + z(x_1 + c_1 z - c_1^2 x_1 + u) \end{aligned}$$

5. To set $\dot{V}_a(x, z) \leq 0$ for each x, z with equality if and only if $x = z = 0$, one possibility is to choose u such that:

$$\begin{aligned} x_1 + c_1 z - c_1^2 x_1 + u &= -c_2 z, \\ \Rightarrow u &= -(c_1 + c_2)z + (c_1^2 + 1)x_1, \end{aligned}$$

for some $c_2 > 0$.

In this case, $\dot{V}_a = -c_1 x_1^2 + c_2 z^2 \leq 0$, with equality if and only if $x_1 = z = 0$. Thus, $(x, z) = (0, 0)$ is globally asymptotically stable. Moreover, $x_2 = z - c_1 x_1$, so (x_1, x_2) is globally asymptotically stable.

6. In terms of the original coordinate system (x_1, x_2), we have:

$$\begin{aligned} V_a(x_1, x_2) &= \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + c_1 x_1), \\ u(x_1, x_2) &= -(1 + c_1 c_2)x_1 - (c_1 + c_2)x_2. \end{aligned}$$

Example. Consider the non-linear system:

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2, \\ \dot{x}_2 &= u. \end{aligned}$$

Use backstepping to find a suitable control u to stabilize the system.

Remark. The linearization of Σ about the origin and that of the linear system in the previous system are the same:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x.$$

Thus, any feedback control that stabilizes the linear system in the previous example also *locally* stabilizes the non-linear system in this example. However, the extra non-linaer curve "x_1^2" term guarantees that the region of attraction R is bounded, and that there exists a finite escape time for the system since it exits R.

Solution :

1. Treating x_2 as a virtual control, choose:

$$x_2 = -c_1 x_1$$

for some $c_1 > 0$, with corresponding closed-loop system:

$$\begin{aligned} \dot{x}_1 &= -c_1 x_1, \\ \dot{x}_2 &= u. \end{aligned}$$

2. Define the error state:

$$z \equiv x_2 - (-x_1^2 - c_1 x_1) = x_2 + x_1^2 + c_1 x_1$$

Then, as $t \to \infty$, we have $x_2 \to -c_1 x_1$ if and only if $z \to 0$.

3. Rewriting the dynamics in terms of x_1, z, we have:

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_1 = x_1^2 + z - x_1^2 - c_1 x_1 \\ &= z - c_1 x_1, \\ \dot{z} &= \dot{x}_1 + 2x_1\dot{x}_1 + c_1\dot{x}_1 = \dot{x}_2 + (2x_1 + c_1)(z - c_1 x_1) \\ &= u + (2x_1 + c_1)(z - c_1 x_1). \end{aligned}$$

4. Define the augmented CLF $V_a(x_1, z)$ by:

$$\begin{aligned} V_a &= V_1(x_1) + \frac{1}{2}z^2 = \frac{1}{2}x_1^2 + \frac{1}{2}z^2, \\ \Rightarrow \dot{V}_a &= x_1\dot{x}_1 + z\dot{z} \\ &= x_1(z - c_1x_1) = z[u + (2x_1 + c_1)(z - c_1x_1)]. \end{aligned}$$

5. To stabilize the system, choose u such that:

$$\begin{aligned} u + (2x_1 + c_1)(z - c_1x_1) &= -c_2z, \\ \Rightarrow u &= -c_2z - (2x_1 + c_1)(z - c_1x_1), \end{aligned}$$

   where $c_2 > 0$.

   In this case, $\dot{V}_a = -c_1x_1^2 + c_2z^2 \leq 0$, with equality if and only if $x_1 = z = 0$. Thus, $(x, z) = (0, 0)$ is globally asymptotically stable. Moreover, $x_2 = z + x_1^2 + c_1x_1$, so (x_1, x_2) is globally asymptotically stable.

6. In terms of the original coordinate system (x_1, x_2), we have:

$$\begin{aligned} V_a(x_1, x_2) &= \frac{1}{2}x_1^2 + \frac{1}{2}(x_1^2 + c_1x_1 + x_2)^2, \\ u(x_1, x_2) &= -(1 + c_1c_2)x_1 - (c_1 + c_2)x_2. \end{aligned}$$

Remark. Backstepping can be applied to robotic systems. For instance, consider the following robotic system, where the position of a robotic arm $x(t)$ is controlled by an input torque $\xi(t)$. The torque is in turn controlled by some input current u.

$$\Sigma : \qquad \begin{aligned} \dot{x} &= f(x) + g(x)\xi, \\ \dot{\xi} &= u. \end{aligned}$$

April 2nd, 2019

Next, we will apply backstepping to general k-dimensional systems. To do so, we will require the definition of "strict feedback." However, we first give the following assumption.

Assumption A1: Below, a system of the form:

$$\Sigma : \ \dot{x} = f(x) + g(x)u,$$

is said to satisfy Assumption A1 if the following statements hold:

1. f, g are locally Lipschitz, with $f(0) = 0$.

2. There exists some $u = \alpha(\cdot) \in C^1$ such that $\alpha(0) = 0$.

3. There exists some $V : \mathbb{R}^n \to \mathbb{R}$, where $V \in C^1$ is positive definite, is radially unbounded, and satisfies:
$$L_f V(x) + L_g V(x)\alpha(x) \leq -W(x)$$
for some positive definite $W(x)$.

Remark. Imposing Assumption A1 on a system Σ allows us to make the following conclusions about Σ:

1. Solutions to Σ exist, are locally unique, and are globablly bounded when $t \geq 0$.
2. $x = 0$ is an equilibrium point of Σ that is stable in the sense of Lyapunov.
3. Since $\dot{V}$ converges to 0, $\lim\limits_{t\to\infty} W(x(t)) = 0$.
4. If the set of trajectories:
$$Z \equiv \{x(\cdot)|W(x(\dot{})) = 0\}$$
contains only the trajectory $x(\cdot) = 0$, then $x = 0$ is globally asymptotically stable.

Definition 3.10 (Strict Feedback). *A system Σ of the following form that satisfies Assumption A1 is said to be a* ***strict feedback system****:*

$$\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)\xi_1 \\
\dot{\xi}_1 &= f_1(x, \xi_1) + g_0(x, \xi_1)\xi_2 \\
\dot{\xi}_2 &= f_2(x, \xi_1, \xi_2) + g_0(x, \xi_1, \xi_2)\xi_3 \\
&\vdots \\
\dot{\xi}_k &= f_k(x, \xi_1, \ldots, \xi_k) + g_0(x, \xi_1, \ldots, \xi_k)u,
\end{aligned}$$

where $f_0(0) = f_1(0,0) = \cdots = f_k(0,0,\cdots,0) = 0$ and $g_0(0) = g_1(0,0) = \cdots = g_k(0,0,\cdots,0) = 0$.

The lemma below formally states the algorithm we applied for backstepping in previous examples.

Lemma 3.11. *Suppose the system $\Sigma : \dot{x} = f(x) + g(x)u$ satisfies Assumption A1, and consider the following augmentation:*

$$\Sigma : \qquad \begin{cases} \dot{x} = f(x) + g(x)\xi, \\ \dot{\xi} = u. \end{cases}$$

1. If $W(x)$ is positive definite, then:
$$V_a(x, \xi) = V(x) + \frac{1}{2}\big(\xi - \alpha(x)\big)^2$$
is a control Lyapunov function for Σ_a.

2. *If $W(x)$ is positive semi-definite (but not necessarily positive definite), then there exists some feedback ξ that renders $\dot{V}$:*

Proof.

1. The proof can be completed by retracing the steps taken in the backstepping examples above.

   (a) Define the difference between the applied control ξ and ideal control $\alpha(x)$ as:

   $$z \equiv \xi - \alpha(x).$$

   (b) Rewrite Σ_a, currently in terms of (x, ξ), in terms of (x, z):

   $$\begin{aligned} \dot{x} &= f(x) + g(x) \cdot (z + \alpha(x)), \\ \dot{z} &= \dot{\xi} - \frac{\partial \alpha}{\partial x}\dot{x} = u - \frac{\partial \alpha}{\partial x}\big[f(x) + g(x) \cdot (z + \alpha(x))\big] \end{aligned}$$

   (c) Define the augmented CLF as:

   $$\begin{aligned} \dot{V}_a &= \underbrace{\frac{\partial V}{\partial x}\Big[f(x) + g(x)\big(z + \alpha(x)\big)\Big]}_{\equiv \dot{V}(x)} + z\underbrace{\left\{u - \frac{\partial \alpha}{\partial x}\Big(f(x) + g(x)\big(z + \alpha(x)\big)\Big) + \frac{\partial V}{\partial x} g(x)\right\}}_{\equiv -cz, \text{ for some } c>0} \\ &= \dot{V}(x) - cz^2 \\ &\leq -W(x) - cz^2 \end{aligned}$$

   The first underbrace in the final line of the above proof follows from the definition of $\dot{V}(x)$; the second indicates that the proof is completed by choosing u such that:

   $$\left\{u - \frac{\partial \alpha}{\partial x}\Big(f(x) + g(x)\big(z + \alpha(x)\big)\Big) + \frac{\partial V}{\partial x} g(x)\right\} = -cz$$

   for some $c > 0$.

2. The proof for this part of the theorem follows similarly.

■

Example. Consider the nonlinear system:

$$\Sigma : \begin{cases} \dot{x} = x\xi, \\ \dot{\xi} = u. \end{cases}$$

We consider the following two approaches.

1. Choose the desired ξ as $\alpha(x) = -x^2$, with CLF given by:

$$V(x) = \frac{1}{2}x^2,$$
$$\Rightarrow \dot{V}(x) = x\dot{x}x(-x^3) = -x^4.$$

The augmented CLF $V_a(x,\xi)$ and corresponding control would then be:

$$V_a(x,\xi) = \frac{1}{2}x^2 + \frac{1}{2}(\xi + x^2)^2,$$
$$\Rightarrow u = -c(\xi + x^2) + 2x \cdot x\xi - x^2.$$

2. Choose the desired ξ as $\alpha(x) = -x^2$, with CLF given by:

$$V(x) = \frac{1}{2}x^2,$$
$$\Rightarrow \dot{V}(x) = x\dot{x} = -x^4.$$

The augmented CLF $V_a(x,\xi)$ and corresponding control would then be:

$$V_a(x,\xi) = \frac{1}{2}x^2 + \frac{1}{2}\xi^2,$$
$$\Rightarrow u = -c\xi - x^2.$$

Since $V_a(\xi,\xi)$ is positive *semi*-definite, it becomes slightly trickier to verify whether the origin is asymptotically stable. For this purpose, we must leverage LaSalle's Theorem. Observe that:

$$S \equiv \{(x,\xi)|V_a(x,\xi) = 0\} = \{(x,\xi)|\xi \equiv 0\} = \{(0,0)\},$$

where the last equality follows from the fact that, if $\xi \equiv 0$, then:

$$x^2 = -u - c\xi = -\dot{\xi} - c\xi = 0,$$

so $x \equiv 0$.

April 4th, 2019

Sliding Mode Control:

Below, we will discuss sliding mode control and the variable structure system.

Example. Consider the system given by $\dot{y} = -ky$, or equivalently, by the first-order differential equation:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -kx_2.$$

Our goal is to design $u(x)$ such that $(0,0)$ is asymptotically stable. First, observe that by re-arranging the above terms, we have:

$$dx_1 = x_2 dt,$$
$$dx_2 = -kx_1 dt,$$
$$\Rightarrow -kx_1 dx_1 = x_2 dx_2,$$
$$\Rightarrow Kx_1^2 + x_2^2 = c,$$

for some constant c. In particular:

- If $K = 1$, the resulting trajectory of the system would be a circle. The trajectories would traverse the circle clockwisely.
- If $K = -1$, the resulting trajectory of the system would be a hyperbola. The trajectories would traverse the circle in the figure **INCOMPLETE; insert figure number below** shown below. **INCOMPLETE; insert figure below**

The central idea of sliding mode control is as follows. Consider the line:

$$s = ax_1 + x_2,$$

where $a > 0$, and let the factor K alternate between the two values $+1$ and -1 in the following manner:

$$K = sgn(sX_1).$$

Then, as Figure **INCOMPLETE, insert figure number below** shows below, the solution "slides" to 0. This is intuitively due to the fact that the applied control pushes the system to the **sliding surface** $s = 0$, at which point it starts oscillating about the sliding surface ("**chattering**") in a zig-zag path towards the origin.

The following example is slightly more general.

Example. Consider the system given by:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = h(x) + g(x) \cdot u,$$

where h, g are unknown, but there exists some $g_0 > 0$ such that $g(\cdot)$ satisfies:

$$g(x_1, x_2) > g_0 > 0$$

for all $x_1, x_2 \in \mathbb{R}$.

Our goal is to design $u(x)$ such that $(0,0)$ is asymptotically stable. Consider the two scenarios below:

- If $u(\cdot)$ is designed to drive x to $s = 0$, then, on S, we would have the dynamics:

$$\dot{x} = x_2 = -ax_1,$$

so $x_1 = 0$ would be asymptotically stable.

- If x is not on $s = 0$, then:

$$\dot{S} = ax_2 + h(x) + g(x) \cdot u.$$

In this case, if there exists some $\sigma(x)$, such that g, h satisfy:

$$\frac{|ax_2 + h(x)|}{g(x)} \leq \sigma(x)$$

for each $x \in \mathbb{R}^2$, then the Lyapunov function becomes:

$$\begin{aligned} V(x) &= \frac{1}{2}s^2, \\ \Rightarrow \dot{V} &= s\dot{s} = s[ax_2 + h(x) + g(x) \cdot u] \\ &\leq |s|g(x) \cdot \sigma(x) + sg(x)u \end{aligned}$$

Now, let a function $\beta(x)$ and some $\beta_0 > 0$ be given such that $\beta(x) \geq \sigma(x) + \beta_0$, and set the control as follows:

$$u(x) = -\beta(x) + \text{sgn}_1(s),$$

where sgn_1 is the *modified sign function* that returns 1, 0, or -1 depending on whether the input argument is positive, zero, or negative, respectively. Then the above equation becomes:

$$\begin{aligned} \dot{V} &\leq |s| \cdot g(x)\sigma(x) - \text{sgn}(x) \cdot g(x)\beta(x) = |s| \cdot g(x)\big[\sigma(x) - \beta(x)\big] \\ &\leq -\beta_0|s| \cdot g(x) = -\sqrt{2}\beta_0 g(x)\sqrt{s} \leq 0, \end{aligned}$$

with equality if and only if $s = 0$.

Remark.

- The function $\sigma(x)$ is very much application-dependent; in particular, it depends on the functions f, g that characterize the given non-linear system.
- The main advantage of sliding mode control is its robustness, while its main disadvantages include constant need for switching and chattering.

April 9th, 2019

Feedback Linearization:

Next, we will discuss feedback linearization, a precise method through which state feedback is applied to a system to make it linear. Consider the following example.

Example. Suppose we wish to stabilize the system:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -a\sin x_1 - bx_2 + cu.$$

using feedback linearization. To linearize the system, set u equal to:

$$u = \frac{1}{c}(a\sin x_1 + v).$$

Then the system becomes:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -bx_2 + v.$$

Now, choose $v = -k_1x_1 - k_2x_2$ to stabilize the system.

Our main question is—Can be generalize this process to any non-linear system? For some systems, this process is directly applicable; for instance, if the given system is of the form:

$$\dot{x} = Ax + B\gamma(x)\cdot[u - \alpha(x)], \tag{3.1}$$

with $\gamma(x) \neq 0$, the control $u = \alpha(x) + \frac{1}{\gamma(x)}v$ stabilizes the system.

The example below indicates that, (1) this procedure is not directly applicable to at least some systems, and that (2) in such cases, there may exist some coordinate transform that renders this procedure applicable.

Example. Consider the system:

$$\dot{x}_1 = a\sin x_2,$$
$$\dot{x}_2 = -x_1^2 + u.$$

Now, suppose we wish to cast the system into the form given by (3.1). Our only choice, shown below, fails to recast the system into a form in which there exists an input that can simultaneously linearize all the given states:

$$\dot{x} = Ox + I\cdot\left(\begin{bmatrix}0\\u\end{bmatrix} - \begin{bmatrix}-a\sin x_1\\x_1^2\end{bmatrix}\right)$$

In particular, the control u is unable to affect the evolution of the first coordinate, given by $\dot{x}_1$, in any way. This implies that we need a coordinate transformation which either renders the dynamics of one coordinate linear without requiring the application of any control, or renders the dynamics in such a form that a single input is sufficient to linearize both coordinates. To that end, consider the effect of applying the coordinate transformation $z_1 = x_1$, $z_2 = a\sin x_2$ back to the original dynamics:

$$\begin{aligned}\dot{z}_1 &= z_2,\\ \dot{z}_2 &= a\cos x_2\cdot(-x_1^2 + u)\\ &= a\cos\left(\sin^{-1}\left(\frac{1}{a}z_2\right)\right)\cdot(-z_1^2 + u).\end{aligned}$$

The control $u = x_1^2 + \frac{1}{a \cos x_2} v$ thus stabilizes the system. We often require the coordinate transformation to be bijective, so as to ensure that the transformed dynamics maintain a clear correspondence to the original dynamics. In that case, we must constrain x_2 to lie within the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Definition 3.12 (Feedback Linearizable). *A nonlinear system of the form:*

$$\Sigma : \qquad \dot{x} = f(x) + g(x)u,$$

*where f, g are sufficiently smooth (depending on u), is called **feedback linearizable** if there exists some control $\overline{u} = \alpha(x) + \beta(x)v$, and some diffeomorphism T such that the change of coordinates $z = T(x)$ and the application of $\overline{u}$ transform Σ to a linear and controllable form, i.e. to the form:*

$$\dot{z} = Az + Bv$$

where (A, B) is controllable.

INSERT FIGURE

Our next questions thus become:

- What are sufficient conditions on f, g that guarantee Σ to be feedback linearizable?
- If Σ is feedback linearizable, what choices of $T, \alpha(\cdot), \beta(\cdot)$ should be chosen to render the closed-loop system both linear and controllable?

First, we consider the tracking problem below. Suppose, for some fixed $y^{des} \in \mathbb{R}$, we wish to choose a control u such that $\lim_{t\to\infty} y = y^{des}$.

Example. Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1^2 + u, \\ y &= x_2. \end{aligned}$$

This is the same system as given in a previous example, modified to include an output y that we wish to steer to y^{des}. The transformation from earlier linearizes the state dynamics, at the cost of rendering the input-output relationship (i.e. between u and y) to become non-linear.

Thus, to allow the system to have a linear input-output relationship, we require an alternate choice of coordinate transformation. For instance, the input $u = x_1^2 + v$ will recast the given system into the following form:

$$\begin{aligned} \dot{x}_1 &= a \sin x_2, \\ \dot{x}_2 &= v, \\ y &= x_2, \end{aligned}$$

which has a linear input-output relationship.

Now, we wish to determine the subset of non-linear systems with given outputs that are input-output linearizable. Consider the single-input-single-output (SISO) system:

$$\dot{x} = f(x) + g(x)u,$$
$$y = h(x),$$

where f, g, h are sufficiently smooth. Then the dynamics of the output y are given by:

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] = L_f h + L_g h \cdot u$$

If $L_g h(x) \neq 0$, then the input choice $u = \frac{1}{L_g h}(-L_f V + v)$ renders the original system input-output linearizable (in particular, with the form $\dot{y} = v$). On the other hand, if $L_g h(x) = 0$, then the first-order evolution of y (characterized by $\dot{y}$) has no direct dependence on the input u. However, it is still a function of x, which evolves under the influence of u, as dictated by the dynamics $\dot{x} = f(x) + g(x)u$. Thus, we may be able to find an explicit dependence between the evolution of y and the input u by consider higher-order terms in the dynamics of y. To that end, consider the following expression for $\ddot{y}$:

$$\ddot{y} = \frac{\partial L_f h}{\partial x}[f(x) + g(x)u] = L_f^2 h + L_g L_f h u.$$

Repeating the above process given for $\dot{y}$, we find that if $L_g L_f h \neq 0$, there exists a control, namely $u = \frac{1}{L_g L_f h}(-L_f^2 h + v)$, that renders the input-output dynamics linear (in particular, of the form $\ddot{y} = v$). If $L_g L_f h = 0$, we must differentiate the dynamics once more:

$$y^{(3)} = \frac{\partial}{\partial x} L_f^2 h(x)[f(x) + g(x)u] = L_f^3 h(x) + L_g L_f^2 h(x) u,$$

and the process continues.

In summary, if for some $r \geq 2$, the function $h(x)$ satisfies:

$$L_g L_f^i h(x) \equiv 0, \qquad \forall i = 0, 1, \cdots, r-2,$$
$$L_g L_f^{r-1} h(x) \neq 0,$$

then, since:

$$y^{(r)} = L_f^{(r)} h(x) + L_g L_f^{(r-1)} h(x) u,$$

the input choice:

$$u = \frac{1}{L_g L_f^{(r-1)} h(x)}[-L_f^r h(x) + v]$$

renders the input-output relationship linear (in particular, of the form $y^{(r)} = v$).

April 11th, 2019

Definition 3.13 (Relative Degree). *An n-dimensional nonlinear system:*

$$\Sigma : \qquad \begin{cases} \dot{x} = f(x) + g(x)u, \\ y = h(x) \end{cases}$$

*is said to have **relative degree** r at $x_0 \in \mathbb{R}^n$ if there exists some $\epsilon > 0$ such that:*

1. *$L_g L_f^r h(x) = 0$, for each $k \in [0, r-1)$, $x \in N_\epsilon(x_0)$.*
2. *$L_g L_f^{r-1} h(x) \neq 0$ for each $x \in N_\epsilon(x_0)$.*

Remark. If the system above is of relative degree r, then $y, \dot{y}, \ddot{y}, \cdots, y^{(r=1)}$ are independent of u, and:

$$y^{(r)} = L_f^r h(x) + L_g L_f^{(r-1)} h(x) u.$$

In this case, choose:

$$u = \frac{1}{L_g L_f^{(r-1)} h(x)} \big[- L_f^r h(x) + v \big]$$

Consider the following examples.

Example. Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_1, \\ \dot{x}_2 &= x_2 + u, \\ y &= x_1. \end{aligned}$$

Then $y^{(r)} = x_1$ does not depend on u, for each $r \in \mathbb{N}$. Thus, the relative degree of this system is not well-defined.

Example. Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_1 + u, \\ \dot{x}_2 &= x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2, \\ y &= x_3. \end{aligned}$$

Differentiating y, we have:

$$\begin{aligned} \dot{y} &= \dot{x}_3 = x_1 x_2, \\ \ddot{y} &= x_1 \dot{x}_2 + x_2 \dot{x}_1 \\ &= x_1(x_2 - x_1 x_3) + x_2(x_1 + u). \end{aligned}$$

Since $\dot{y}$ does not explicitly depend on u, but $\ddot{y}$ does, the system has depth 2 in $D \equiv \{x \in \mathbb{R}^3 | x_2 \neq 0\}$.

The next theorem below shows that, if a nonlinear system with output has relative degree (exactly) equal to its dimension, it is feedback linearizable. Before doing so, however, we require the following two facts.

Theorem 3.14 (**Inverse Function Theorem**). *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable, with a non-singular Jacobian, i.e. with:*

$$\det\left(\frac{\partial J}{\partial x}(x_0)\right) = 0.$$

Then T^{-1} exists and is continuously differentiable. In other words, T is diffeomorphic.

Theorem 3.15 (**Part of Theorem 4.2.3. in [11]**). *If the nonlinear system:*

$$\Sigma : \qquad \begin{cases} \dot{x} = f(x) + g(x)u, \\ y = h(x) \end{cases}$$

has relative degree r at x_0, then there exists a neighborhood of x_0 in which:

$$\text{rank}\left(\frac{\partial h}{\partial x}\frac{\partial L_f h}{\partial x} \vdots \frac{\partial L_f^{(r-1)}}{\partial x}\right) = r.$$

Armed with the above facts, we can now state the following.

Theorem 3.16. *An n-dimensional nonlinear system:*

$$\Sigma : \qquad \begin{cases} \dot{x} = f(x) + g(x)u, \\ y = h(x) \end{cases}$$

of relative degree n is feedback linearizable.

Proof. Suppose the nonlinear system Σ, as given above, has relative degree n. Then:

$$\begin{aligned} y^{(k)} &= L_f^{(k)} h(x), \qquad k = 1, \cdots, n-1, \\ y^{(n)} &= L_f^{(n)} h(x) + L_g L_f^{(n-1)} u. \end{aligned}$$

Then, with the control input choice:

$$u = \frac{1}{L_g L_f^{(n-1)} h(x)} \left[- L_f^n h(x) + v \right]$$

we have $y^{(n)} = v$. Now, define $z = (y, \dot{y}, \cdots, y^{(n-1)})^T \in \mathbb{R}^n$. Then:

$$\dot{z} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\equiv A} v + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\equiv B} v.$$

Observe that (A, B) is controllable (in fact, it is in the controllable canonical form). Moreover, Theorem 3.15 implies that the mappping $x \mapsto z$ is diffeomorphic. Thus, any nonlinear system of relative degree equal to its dimension is feedback linearizable. ■

Our next question thus becomes—When does there exist an output $h(x)$ of relative degree n? To answer this question, we must leverage the following definitions from differential geometry—manifolds, tangent space, tangent vector, tangent bundle, distribution, Lie brackets, involute distribution.

April 16th, 2019

Differential Geometry, 2019

Prediction

Below, we begin a brief introduction of differential topology. Most of the following definitions are reoragnized from Section 3.9 of [10].

Definition 3.17 (**Smooth**)**.** *Let $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ be arbitrary subsets of $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively. A mapping $f : X \to Y$ is said to be* ***smooth*** *on some open set $U \subset X$ if all of its partial derivatives, i.e.:*

$$\frac{\partial^n f}{\partial x_{i_1}}, \cdots, \frac{\partial^n f}{\partial x_{i_n}}$$

for each $n \in \mathbb{N}$, exist and are continuous in U.

Theorem 3.18. *If $f : X \to Y$ and $g : Y \to Z$ are smooth, then so is $g \circ f : X \to Z$.*

Definition 3.19 (**Homeomorphism, Diffeomorphism**)**.** *Let $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ be arbitrary subsets of $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively, and let $f : X \to Y$ be given.*

1. *f is called a* ***homeomorphism*** *if f is continuous, is bijective, and has a continuous inverse.*

2. *f is called a* ***diffeomorphism*** *if f is continuous, is bijective, and has a smooth inverse.*

Remark. A diffeomorphism is essentially a homeomorphism with a smooth inverse.

Definition 3.20 (**Smooth Manifold of Dimension** m)**.**

1. *A subset $M \subset \mathbb{R}^k$ is called a* ***smooth manifold of dimension*** *m if, for each $x \in M$, there exists a neighborhood $W \subset \mathbb{R}^k$ of x such that $W \cap M$ is diffeomorphic to an open subset of $\mathbb{R} \subset \mathbb{R}^m$.*

2. *A diffeomorphism $\psi : W \cap M \to U$ is called a* ***system of coordinates*** *on $W \subset M$, and its inverse $\psi^{-1} : U \to W \cap M$ is called a* ***parameterization****. The mappings are referred to as* ***coordinate maps****.*

An alternative definition for smooth manifolds eschews the above construction via coordinate maps for a characterization via the Inverse Function Theorem. (This is the definition given in the class notes).

Definition 3.21 (**Smooth Manifold**). *Let $M \subset \mathbb{R}^n$ be a non-empty subset of $\mathbb{R}^n$, and fix $m \in \{1, \cdots, n\}$. Then M is an m-dimensional smooth manifold of $\mathbb{R}^n$ if, for each $p \in M$, there exists some $r > 0$ and smooth $F : B_r(p) \to \mathbb{R}^{n-m}$ such that:*

1. *For each $q \in M \subset B_r(p)$, we have $F(q) = 0$, and*
2. *For each $q \in M \cap B_r(p)$:*
$$\text{rank}\left(\frac{\partial F}{\partial x}(q)\right) = n - m.$$

In other words, a manifold is the zero level set of some smooth function F whose derivative satisfy certain rank conditions.

Some examples (and non-examples) of manifolds are given below.

Example.

1. The unit circle $S^1 \subset \mathbb{R}^2$, defined by the diffeomorphism:
$$f(\theta) = (\cos\theta, \sin\theta)$$
for each $\theta \in [0, 2\pi)$, is a smooth manifold of dimension 1. Equivalently, it is the zero-level set of the function:
$$F(x_1, x_2) = x_1^2 + x_2^2 - 1.$$
2. The unit circle S on the x-y plane in $\mathbb{R}^3$ is the zero-level set of the function:
$$F(x_1, x_2, x_3) = (x_1^2 + x_2^2 - 1, x_3)$$
3. The unit sphere $S^2 \subset \mathbb{R}^3$, piecewisely defined by the diffeomorphisms:
$$f_1(x_1, x_2) = (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2},$$
$$f_2(x_1, x_2) = (x_1, x_2, -\sqrt{1 - x_1^2 - x_2^2},$$
for each (x, y) satisfying $x^2 + y^2 < 1$, is a smooth manifold of dimension 2. More generally, the n-sphere $S^n \subset \mathbb{R}^{n+1}$ is given by:
$$S^n = \{x \in \mathbb{R}^{n+1} | \|x\|_2 = 1\}.$$
Equivalently, S^2 is the zero-level set of the function:
$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_2^3 - 1.$$

4. A cone in $\mathbb{R}^3$, given by:

$$f(x_1, x_2) = (x_1, x_2, \sqrt{x_1^2 + x_2^2}$$

is *not* a manifold of dimension 2, since there is no diffeomorphism that maps a neighborhood of the vertex of the cone onto an open subset of $\mathbb{R}^2$.

5. Let $SO(2)$ denote the space of orthogonal matrices in $\mathbb{R}^{2\times 2}$ with determinant 1, i.e.:

$$SO(2) \equiv \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \middle| \theta \in [0, 2\pi) \right\}$$

Then $SO(2)$ is a manifold of dimension 1. In fact, since the matrix is periodic in θ, the set $SO(2)$ is diffeomorphic to S^1.

Next, we develop tools for performing calculus on manifolds.

Definition 3.22 (Tangent Space). *Let M be an m-dimensional smooth manifold in $\mathbb{R}^n$, and fix $p \in M$. Then there exists some smooth function $F : B_r(p) \to \mathbb{R}^{n-m}$ satisfying the above definition of a tangent space. The **tangent space to** M **at** p is thus defined by:*

$$T_p(M) = N\left(\frac{\partial F}{\partial x}(p)\right),$$

*where T_p denotes the tangent plane. Observe that, by definition of F, we have $\dim(T_pM) = m$. Vectors in $T_p(M)$ are said to be **tangent vectors to** M **at** p.*

Definition 3.23 (Vector Field). *Given an m-dimensional manifold M in $\mathbb{R}^n$, a **vector field on** M is a mapping $f : M \to T_p(M)$ that assigns each $p \in M$ to some tangent vector $f(p) \in M$. The vector field is said to be C^k if f is C^k.*

Remark. In the context of a nonlinear system, if:

$$\dot{x} = f(x) + \sum_{i=1}^{k} g_i(x)u_i,$$

where $u_1, \cdots, u_k$ are scalars, then $f, g_1, \cdots, g_k$ are vector fields.

Definition 3.24 (Sub-manifold, Invariant sub-manifold). *Suppose M is a smooth manifold, and f is a locally Lipschitz vector field on M.*

1. *If $N \subset M$ is itself a manifold, it is said to be a **sub-manifold** of M.*

2. *If, for any $x_0 \in M$, the solution to:*

$$\dot{x} = f(x), \qquad x(t_0) = x_0$$

*lies in M for all $t \geq t_0$, then $N \subset M$ is called an **invariant submanifold** of M.*

Definition 3.25 (**Lie Bracket**). *Given two vector fields f, g, the Lie Brackets is defined by:*

$$[f,g](x) \equiv \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) = (L_f g - L_g f)(x)$$

For multiple applications of the Lie Bracket, we use the ***adjoint notation****, given by:*

$$\begin{aligned} \mathrm{adj}_f^0 g(x) &\equiv g(x), \\ \mathrm{adj}_f^k g(x) &\equiv [\mathrm{adj}_f^{k-1} g(x) g(x)], \qquad \forall k \in \mathbb{N}. \end{aligned}$$

In other words, $\mathrm{adj}_f^k g(x) = [f, [f, \cdots, [f, g] \cdots]]$*, with* $n-1$ f*s.*

Remark. The Lie Bracket of f, g can describe a "direction of travel" not given directly by the span of f and g. See the second example below.

Example. If $f(x) = (x_2, -\sin x_1 - x_2), g(x) = (0, x_1)$, then:

$$[f,g](x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix}.$$

Example. Consider the n-dimensional linear system $\dot{x} = Ax + bu$, where $A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n$. Then:

$$\begin{aligned} [f,g] &= -Ab, \\ [f,[f,g]] &= A^2 b, \\ &\vdots \\ [f,[f \cdots, [f,g] \cdots]] &= (-1)^n A^n b. \end{aligned}$$

Definition 3.26 (**Tangent Bundle**). *The* ***tangent bundle*** TM *of a manifold* M *is the union of its tangent spaces, i.e.:*

$$TM \equiv \bigcup_{p \in M} T_p(M).$$

Definition 3.27 (**Distribution**). *Given a manifold* M*, a* ***distribution*** *is the span of a collection of vector fields on* M*, i.e. given vector fields* $f_1, \cdots, f_k$*, defined with respect to a manifold* M*, the distribution associated with* $f_1, \cdots, f_k$ *is defined by:*

$$\Delta(x) = \mathrm{span}\{f_1(x), \cdots, f_k(x)\}$$

Observe that, at each x*, the distribution* $\Delta(x)$ *is a subspace, with dimension given by:*

$$\dim\big(\Delta(x)\big) = \mathrm{rank}\big(\begin{bmatrix} f_1(x) & \cdots & f_k(x) \end{bmatrix}\big)$$

Definition 3.28 (**Non-singular Distribution, Involute Distribution**). *Let* Δ *be a* ***distribution*** *on some manifold* M*.*

1. Δ *is called* ***non-singular*** *if* $\dim\big(\Delta(x)\big)$ *is independent of* x.

2. Δ *is called* ***involutive*** *if, for each* $f, g \in \Delta$ *and* $x \in M$, *we have:*

$$[f, g] \in \Delta.$$

Example. If $f_1(x) = (2x_2, 1, 0)$, $f_2(x) = (1, 0, x_2)$, and $\Delta(x) \equiv \operatorname{span}\{f_1(x), f_2(x)\}$, then:

$$\dim\big(\Delta(x)\big) = \operatorname{rank}\left(\begin{bmatrix} 2x_2 & 1 \\ 1 & 0 \\ 0 & x_2 \end{bmatrix}\right) = 2, \qquad \forall x_2 \in \mathbb{R},$$

$$[f, g] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In fact, $\Sigma : \dot{x} = f(x)$ is feedback linearizable if and only if the state-dependent matrix:

$$\begin{bmatrix} g(x) & \operatorname{ad}_f g(x) & \cdots & \operatorname{ad}_f^{n-1} g(x) \end{bmatrix}$$

is non-singular with rank m for each n, and its range space:

$$\Delta(x) \equiv R\Big(\begin{bmatrix} g(x) & \operatorname{ad}_f g(x) & \cdots & \operatorname{ad}_f^{n-1} g(x) \end{bmatrix} \Big)$$

is involutive.

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Proposition 3.29. *The control affine system:*

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned}$$

has relative degree n *at* $x_0 \in \mathbb{R}^n$, *i.e.* $L_g h^i(x_0) = 0, \forall i = 0, \cdots, n-2$, *and* $L_g h^{n-1}(x_0) \neq 0$, *if and only if any of the following two statements are true:*

$$L_g h(x_0) = L_{ad_f g} h(x_0) = \cdots = L_{ad_f^{n-2} g} h(x_0) = 0, \qquad L_{ad_f^{n-1} g} h(x_0) \neq 0, \tag{3.2}$$

$$\Longleftrightarrow \left.\frac{\partial h}{\partial x}\right|_{x_0} \begin{bmatrix} g(x) & ad_f g(x_0) & \cdots & ad_f^{n-2} g(x_0) \end{bmatrix} = 0 \tag{3.3}$$

The equivalence of (3.2) and (3.3) is left as an exercise to the reader.

Definition 3.30 (Completely Integrable). *A non-singular* k*-dimensional distribution:*

$$\Delta(x) = \operatorname{span}\{f_1(x), \cdots, f_k(x)\}, \qquad \forall x \in \mathbb{R}^n$$

is called ***completely integrable*** *if there exist* $n-k$ *functions* $\phi_1(x),\cdots,\phi_{n-k}(x)$*, such that:*

$$\frac{\partial \phi_i}{\partial x} f_j(x) = 0, \qquad \forall i = 1,\cdots,n-k, j = 1,\cdots,k, \qquad and:$$
$$\left\{ \frac{\partial \phi_i}{\partial x} \middle| i = 1,\cdots,k \right\} \text{ is linearly independent.}$$

Theorem 3.31 (**Frobenius**)**.** *A non-singular distribution is completely integrable if and only if it is involutive.*

Proof. (see pgs. 360-362 of Sastry, Shankar, "Nonlinear Systems" [10]). ■

Theorem 3.32 (**Feedback Linearization**)**.** *An affine control system* $\Sigma : \dot{x} = f(x) + g(x)u$ *is feedback linearizable, for each* $x \in \mathbb{R}^n$*, if and only if the following conditions both hold:*

1. $\text{rank}\Big(\big[g(x) \quad \text{ad}_f g(x) \quad \cdots \quad \text{ad}_f^{n-1} g(x) \big] \Big) = n$ *for each* $x \in \mathbb{R}^n$*.*
2. $\Delta = \text{span}\{g(x), ad_f g(x), \cdots, ad_f^{n-2} g(x)\}$ *is involutive.*

Proof.
" ⇒: " (Omitted)

⇐: " Below, we will show that the above two conditions imply the equalities in (3.2). Observe the first condition implies that $\Delta(x)$ is non-singular with degree $n-1$, while the second indicates that $\Delta(x)$ is involutive. By Frobenius' Theorem, Δ is completely integrable. Taking $k = n-1$ in the definition of complete integrability, we find that there exists some $h(x)$ such that:

$$\frac{\partial h}{\partial x} \text{ad}_f^i g(x) = 0, \qquad j = 1,\cdots,n-1,$$
$$\frac{\partial h}{\partial x} = 0.$$

It remains to show that $L_{\text{ad}_f^{n-1} g} h(x_0) \neq 0$. Suppose by contradiction that:

$$0 = L_{\text{ad}_f^{n-1} g} h(x_0) = \left. \frac{\partial h}{\partial x} \right|_{x_0} \text{ad}_f^{n-1} g(x_0),$$
$$\Rightarrow \left. \frac{\partial h}{\partial x} \right|_{x_0} \big[g(x) \quad ad_f g(x_0) \quad \cdots \quad ad_f^{n-1} g(x_0) \big] = 0,$$
$$\Rightarrow \left. \frac{\partial h}{\partial x} \right|_{x_0} = 0,$$

a contradiction. Observe that the final equality follows from the first statement in the theorem, i.e. the assertion that $rank\Big(\big[g(x) \quad ad_f g(x) \quad \cdots \quad ad_f^{n-1} g(x) \big] \Big) = n$ for each $x \in \mathbb{R}^n$. We have thus established that (3.2) holds, so Σ has relative degree n. This in turn implies that Σ is feedback linearizable. ■

MIMO Feedback Linearization

Definition 3.33 (**Square Affine System**). *A* ***square affine system*** *is a control affine system where the control u and output y have the same number of inputs and outputs, i.e. it has the form:*

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i,$$
$$y = h(x) = (h_1, \cdots, h_m)(x),$$

for some $m \in \mathbb{N}$.

We wish to extend the concept of relative degree from single-input-single-output systems to square affine systems of arbitrary dimensions. Intuitively, this can be done by taking each output h_i (for each $i \in 1, \cdots, n$), and examining the lowest time derivative degree r_i at which $h_i^{(r_i)}$ explicitly depends on some u_j. Moreover, we want each of the inputs u_j to be of use. Thus, the dependence of h_is to be spread out among these u_js, i.e. we cannot have a square affine system each output $h_1, \cdots, h_m$ explicitly depends on u_1, but no output explicitly depends on $u_2, \cdots, u_m$. This intuition is translated into precise mathematical language below.

Definition 3.34 (**Vector Relative Degree**). *A square affine system Σ is said to have* ***vector relative degree*** *$r = (r_1, \cdots r_m) \in \mathbb{R}^m$ at $x_0 \in \mathbb{R}^n$ if both of the following conditions hold. Observe that i, j, k denote indices for the input component, output component, and degree respectively:*

1. $L_{g_j} L_f^k h_i(x) = 0$, *for each* $i, j = 1, \cdots, m$, $k = 1, \cdots, r_i - 1$,
2. $\text{rank}\big(\big[L_{g_j} L_f^{r_i - 1} h_i(x_0)\big]_{ij}\big) = m$,

where $\big[L_{g_j} L_f^{r_i - 1} h_i(x_0)\big]_{ij}$ is called the ***decoupling matrix****.*

Remark. The decoupling matrix directly associates the control signals $u_1, \cdots, u_m$ to the derivatives of the outputs, i.e. to $h_1^{(r_1)}, h_m^{(r_m)}$, as shown below:

$$h_i^{(r_i)} = L_f^{r_i} h_i(x) + \sum_{j=1}^{m} L_{g_j} L_f^{r_i - 1} h_i(x) u_j, \qquad \forall i = 1, \cdots, m,$$
$$\Rightarrow \begin{bmatrix} L_f^{(r_1)} h_1(x) \\ \vdots \\ L_f^{(r_m)} h_m(x) \end{bmatrix} + A(x)u.$$

Thus, if we want $h_i^{(r_i)} = v_i$ for some set of inputs $\{v_i | i = 1, \cdots, m\}$, set:

$$u = A(x)^{-1} \left(- \begin{bmatrix} L_f^{(r_1)} h_1(x) \\ \vdots \\ L_f^{(r_m)} h_m(x) \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \right)$$

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Feedback Linearization—Standard Procedure:

A standard procedure for feedback linearization can thus be given as follows:

1. Construct $\{g, \text{ad}_f g, \cdots, \text{ad}_f^{n-1} g\}$.
2. Check that $\{g, \text{ad}_f g, \cdots, \text{ad}_f^{n-1} g\}$ has full rank (i.e. rank $= n$), and $\Delta = \text{span}\{g(x), \cdots, \text{ad}_f^{n-2} g(x)\}$ is involutable. If this holds in some neighborhood of the equilibrium point, the system is locally feedback linearizable. If this holds everywhere in the state space, the system is globally feedback linearizable.
3. Find an output $h(x)$ with relative degree n.
4. Construct appropriate input and state transformations based on the definition of $h(x)$.

The following example shows these four steps in action.

Example. (**Feedback Linearization Example**) Consider the 2-dimensional non-linear system:

$$\Sigma: \qquad \dot{x} = \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Below, we apply the four steps described above.

1. First, find g and $\text{ad}_f g$:

$$\begin{aligned} g &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ \text{ad}_f g = [f, g] &= \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \\ &= - \begin{bmatrix} 0 & a \cos x_2 \\ -2x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \cos x_2 \\ 0 \end{bmatrix} \end{aligned}$$

2. We wish to find the rank of $\begin{bmatrix} g & \text{ad}_f g \end{bmatrix}$ and the involutability of g:

$$\begin{aligned} \begin{bmatrix} g & \text{ad}_f g \end{bmatrix} &= \begin{bmatrix} 0 & -a \cos x_2 \\ 1 & 0 \end{bmatrix}, \qquad \forall x_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \dim\big(\text{span}\{g\}\big) &= 1, \qquad \forall x_2 \in \mathbb{R}. \end{aligned}$$

3. Next, we wish to find $h(x)$ such that $h(x)$ has relative degree 2, i.e. $L_g h(x) = 0$ and $L_g L_f h(x) \neq 0$. For simplicity, we will use the following abbreviations:

$$\begin{aligned} h_i &\equiv \frac{\partial h}{\partial x_i}, \qquad \forall i = 1, 2, \\ h_{i,j} &\equiv \frac{\partial^2 h}{\partial x_i x_j}, \qquad \forall i, j = 1, 2, \end{aligned}$$

We thus have:

$$
\begin{aligned}
0 = L_f h(x) &= \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0, \\
0 \neq L_g L_f h(x) &= L_g \left(\begin{bmatrix} h_1 & 0 \end{bmatrix} \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} \right) = L_g \left(a \sin x_2 h_1 \right) \\
&= \begin{bmatrix} \frac{\partial}{\partial x_1}(a \sin x_2 h_1) & \frac{\partial}{\partial x_2}(a \sin x_2 h_1) \end{bmatrix} \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial}{\partial x_1}(a \sin x_2 h_{11}) & a \cos x_2 h_1 + a \sin x_2 h_{22} \end{bmatrix} \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} \\
&= a^2 \sin^2 x_2 h_{11} - a x_1^2 \cos x_2 h_1 - a x_1^2 \sin x_2 h_{12}
\end{aligned}
$$

In particular, the first constraint tells us that $h(x) = h(x_1, x_2)$ is independent of x_2, while the second constraint imposes additional restrictions. A possible choice is $h(x) = x_1$.

4. Rewrite the dynamics using h and $\dot{h}$, and find the corresponding input and state transformations.

The state transformation is given by:

$$
\begin{aligned}
y &= x_1, \\
\dot{y} &= a \sin x_2, \\
\ddot{y} &= -a(x_1^2 + u) \cos x_2,
\end{aligned}
$$

while the input transformation is given by:

$$
u = -x_1^2 - \frac{v}{a \cos x_2}
$$

Input-Output Linearization, SISO Case:

Below, we motivate the definition of the zero dynamics of a system using the following example.

Example. (**Input-Output Linearization Example 1, SISO Case**) Consider the following system:

$$
\Sigma_1 : \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 + u \\ -u \end{bmatrix}, \qquad y = x_1.
$$

Our goal is as follows—For a given desired output $y_d(\cdot)$, choose an input $u(\cdot)$ such that $e \equiv y - y_d \to 0$ as $t \to \infty$, while keeping the state vector x bounded.

To examine what inputs u allow $e \to 0$, we must evaluate the dynamics of e:

$$
\dot{e} = \dot{y} - \dot{y}_d(t) = x_2^3 - \dot{y}_d(t) + u.
$$

This indicates that we should choose $u \equiv -x_2^3 - \dot{y}_d(t) - e$. In this case, the internal dynamics become:

$$\dot{x}_2 = x_2^3 + e - \dot{y}_d(t).$$

Suppose $\dot{y}_d(t)$ is bounded, i.e. there exists some $D > 0$ such that $|e(t) - \dot{y}_d(t)| \leq D$. Then, in this case, the above choice of u renders x_2 unbounded, which in turn implies that u itself becomes unbounded. This is because it can be shown that, if $|x_2| > \sqrt[3]{D}$, we have $\text{sign}(\dot{x}_2) = \text{sign}(x_2)$. Thus, $|x_2| \to \infty$, and so $|u| = |-x_2^3 + \dot{y}_d(t) - e| \to \infty$.

In summary, the above analysis shows that no bounded input can reduce the error e to 0.

We wish to find an input that keeps the internal dynamics stable. For linear systems, this is denoted by the locations of the zeros in the system's transfer function. Inspired by this observation, we define the **zero dynamics** of the system to be the state dynamics corresponding to an input choice that keeps the output identically zero.

Proposition 3.35. *Given a nonlinear system Σ, the local (asymptotic) stability of the zero dynamics implies local (asymptotic) stability of the internal dynamics.*

Essentially, the above proposition tells us that, if the internal dynamics are too difficult to analyze, try to analyze the zero dynamics instead.

Example. (**Input-Output Linearization Example 2, SISO Case**) Consider the following system:

$$\Sigma_1 : \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 + u \\ u \end{bmatrix}, \qquad y = x_1.$$

In this case, the zero dynamics are given by:

$$x_1 = 0,$$
$$\dot{x}_1 = x_2^3 + u = 0.$$

Thus, to drive $y \equiv x_1 \to 0$, we must choose $u = x_2^3$. As a result, the dynamics for x_2 become:

$$\dot{x}_2 = -u = -x_2^3,$$

which drives $x_2 \to 0$ asymptotically. Since the zero dynamics of the system is stable, so are the internal dynamics (at least locally).

Input-Output Linearization, MIMO Case:

A standard procedure for feedback linearization can thus be given as follows:

1. Differentiate y until u appears.

2. One control choice is to choose u to cancel non-linearities in the system, thus guaranteeing input/output stability of the resulting linear system.

3. Study the stability of the internal dynamics of the system, through studying the zero dynamics.

   In particular, consider the following MIMO system of vector degree $(r_1, \cdots, r_m)$:

$$\Sigma: \qquad \begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} L_f^{r_1-1} h_1(x) \\ \vdots \\ L_f^{r_m-1} h_m(x) \end{bmatrix} + A(x)u$$

Thus, the following choice of input will stabilize the system:

$$u = A^{-1}(x) \cdot \left[- \begin{bmatrix} L_f^{r_1-1} h_1(x) \\ \vdots \\ L_f^{r_m-1} h_m(x) \end{bmatrix} + u \right]$$

The feedback linearizability of Σ depends on $(r_1, \cdots, r_m)$, as the following proposition demonstrates.

Proposition 3.36. *Suppose a multiple-input-multiple-output non-linear system has vector relative degree $(r_1, \cdots, r_m)$. Then:*

1. *If $r_1 + \cdots + r_m = n$, the system is feedback linearizable.*
2. *If $r_1 + \cdots + r_m < n$, the system may still be input-output linearizable.*

   To find the zero dynamics of a MIMO system, we must find the control u that satisfies:

$$0 \equiv y = \begin{bmatrix} y_1^{(r_1)} \\ \vdots \, y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} L_f^{(r_1)} h_1 \\ \vdots \, L_f^{(r_m)} h_m \end{bmatrix} + A(x) \cdot u$$

This is given by $u^\star$, as defined below, with corresponding dynamics:

$$u^\star \equiv -A(x)^{-1} \begin{bmatrix} L_f^{(r_1)} h_1 \\ L_f^{(r_m)} h_m \end{bmatrix},$$

$$f^\star(x) = f(x) + g(x)u^\star.$$

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Example. Consider the system:

$$\Sigma: \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + u \\ -u \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\equiv A} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\equiv B} u,$$

$$y = x_1 = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\equiv C} x.$$

Design a controller u that drives $y \to y^d$ as $e \to 0$.

Solution :

Define the error $e \equiv y - y^d$, with corresponding error dynamics:

$$\dot{e} = \dot{y} - \dot{y}^d = x_2 + u - \dot{y}^d.$$

In this case, if we choose $u = -e - x_2 + \dot{y}^d$, we have error and internal dynamics given as follows:

$$\begin{aligned} \dot{e} &= -e, \\ \dot{x}_2 &= x_2 + e - \dot{y}^d. \end{aligned}$$

We claim that the zero dynamics of the given linear system are dictated by the zero locations. This can be verified by examining the input-output transfer function of the system, given by:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{s-1}{s^2} \end{aligned}$$

Thus, the system has a zero $s = +1$ in the right-half complex plane. This indicates that the zero dynamics are unstable for this example. This agrees with the fact that the internal dynamics is intuitively unstable, in the sense that:

$$\dot{x}_2 = x_2 + (e - \dot{y}^d)$$

asymptotically approaches the unstable linear system $\dot{x}_2 = x_2$ as $t \to \infty$.

■

Below, we give a more general example of the zero dynamics of a linear system.

Example. Consider a linear system with input-output transfer function:

$$G(s) = \frac{s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

Analyze the zero dynamics of the system.

Solution :

The given system can be rendered into a controllable canonical form:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a_3 & -a_2 & -a_1 & -a_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} b_0 & b_1 & 1 & 0 \end{bmatrix} x. \end{aligned}$$

From the definition of the output y, we have:

$$\begin{aligned} y &= Cx, \\ \dot{y} &= CAx + CBu = CAx, \\ \ddot{y} &= CA^2x + CABu, \\ \iff \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ CAB \end{bmatrix} u, \end{aligned}$$

where we observe that $CB = 0, CAB \neq 0$. The system thus has relative degree 2.

Now, let us consider the zero dynamics of the system. Suppose $y = \dot{y} = 0$. Then:

$$\begin{aligned} 0 = y &= b_0x_1 + b_1x_2 + b_2x_3, \\ \Rightarrow \dot{x}_2 &= x_3 = -b_0x_1 - b_1x_2. \end{aligned}$$

In other words, the zero dynamics are given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -b_0 & -b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

with characteristic polynomial given by $s^2 + b_1s + b_0$.

Example. Next, consider the example of a planar quadrotor, given by:

$$\Sigma : \qquad \begin{aligned} m\ddot{y} &= -F\sin\theta, \\ m\ddot{z} &= F\cos\theta - mg, \\ J\ddot{\theta} &= M. \end{aligned}$$

To render the above dynamics into the form of a first-order differential equation, define the system state $x = (y, z, \theta, \dot{y}, \dot{z}, \dot{\theta})$. Then:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{m}\sin x_3 & 0 \\ \frac{1}{m}\cos x_3 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} F \\ M \end{bmatrix}.$$

If we choose the output to be $y_1 \equiv (x_1, x_2) \equiv h(x) \to 0$, then:

$$\begin{aligned} \dot{y}_1 &= L_fh(x) + L_gh(x)u = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, \\ \Rightarrow \ddot{y}_1 &= L_f^2h(x) + L_gL_fh(x)u = \begin{bmatrix} 0 \\ -g \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin x_1 & 0 \\ \frac{1}{m}\sin x_1 & 0 \end{bmatrix} u. \end{aligned}$$

Observe that $r_1 = r_2 = 2$, but the vector relative degree of the system is not well-defined, since $L_gL_fh(x)$ is not invertible. This implies that this direct feedback scheme does not work.

Specifically, since $u = (F, M)$, the second column of zeros in $L_g L_f h(x)$ implies that applying M has no effect on the dynamics.

To allow M to take effect, we instead make use of dynamic feedback linearization, i.e. a feedback controller that has dynamics of its own:

$$
\begin{aligned}
u &= \alpha(x, \xi) + \beta(x, \xi)u, \\
\dot{\xi} &= r(x, \xi) + \delta(x, \xi)v.
\end{aligned}
$$

The system dynamics thus becomes:

$$
\begin{aligned}
\dot{x} &= f(x) + g(x) \cdot \begin{bmatrix} \xi \\ M \end{bmatrix}, \\
\dot{\xi} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F \\ M \end{bmatrix}
\end{aligned}
$$

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Example. Use sliding mode control to stabilize the following system:

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \sin x_1, \\
\dot{x}_2 &= \theta_1 x_1^2 + (1 + \theta_2)u
\end{aligned}
$$

Solution:

Identify $h(x) = \theta_1 x_1^2$ and $g(x) = 1 + \theta_2$; then $\dot{x}_2 = h(x) + g(x)u$, as in the case of normal sliding mode control. The presence of the additional term "$\sin x_1$" in the expression for $\dot{x}_1$ implies that the expression for s would need to be adjusted slightly to allow trajectories on the sliding surface $s = 0$ to move towards the origin asymptotically. For instance, we can choose s such that, on $s = 0$, we have $\dot{x}_1 = x_2 + \sin x_1 = -ax_1$ for some $a > 0$. This could be done, for instance, by defining:

$$
s = \sin x_1 + ax_2 + x_2
$$

In this case, we have:

$$
\dot{s} = (\cos x_1 + a)(x_2 + \sin x_1) + h(x).
$$

The standard operating procedure for sliding mode control then implies that we must choose some $\sigma(x)$ such that:

$$
\frac{|(\cos x_1 + a)(x_2 + \sin x_1) + h(x)|}{g(x)} \leq \sigma(x).
$$

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